VOLUME LII NUMBER 4

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THE ASTROPHYSICAL JOURNAL

AN INTERNATIONAL REVIEW OF SPECTROSCOPY AND ASTRONOMICAL PHYSICS

EDITED BY

GEORGE E. HALE
Mount Wilson Observatory of the Carnegie Institution of Washington

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NOVEMBER 1920

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THE UNIVERSITY OF CHICAGO PRESS
CHICAGO, ILLINOIS, U.S.A.

THE CAMBRIDGE UNIVERSITY PRESS, London
THE MARUZEN-KABUSHIKI-KAISHA, Tokyo, Osaka, Kyoto, Fukuoka, Sendai
THE MISSION BOOK COMPANY, Shanghai

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Claims for missing numbers should be made within the month following the regular month of publication. The publishers expect to supply missing numbers free only when losses have been sustained in transit, and when the reserve stock will permit.

Business correspondence should be addressed to The University of Chicago Press, Chicago, Illinois.

Communications for the editors and manuscripts should be addressed to the Editors of The Astrophysical Journal, Yerkes Observatory, Williams Bay, Wisconsin.

The cable address is "University, Chicago."

The articles in this Journal are indexed in the *Readers' Guide Supplement*, New York, N.Y.

Entered as second-class matter, January 17, 1895, at the Post-office at Chicago, Ill., under the act of March 3, 1879.

Acceptance for mailing at special rate of postage provided for in Section 1103, Act of October 3, 1917, authorized on July 15, 1918.

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VOLUME LII NOVEMBER 1920 NUMBER 4

PHOTOGRAPHIC SHARPNESS AND RESOLVING POWER[1]

By FRANK E. ROSS

ABSTRACT

Photographic sharpness.—The paper begins by summarizing the more important relations between the fundamental photographic variables. It is found experimentally that for a considerable range of exposures the diameter Δ of an image is $\Delta = a + 2\kappa \log t$; or $\Delta = a' + 2\kappa' \log I$ (though the Greenwich equation $\Delta^{\frac{1}{2}} = a' + b' \log t$ holds over a range 30 times greater). The constant κ is called the turbidity and is very nearly equal to κ', the ratio being Schwarzschild's coefficient p. The author derives the following formula for the intensity at various distances x from the edge of an image: $\log I = \log I_0 - x/\kappa'$, which is merely Beer's law for a homogeneous medium with extinction coefficient $1/\kappa'$; and sharpness, defined as the rate of change of density with x, is shown to be equal to the ratio of the contrast function γ to κ. While actual *measurements of the variation of sharpness with wave-length and with time of development* prove this *theoretical expression for sharpness* to be but a first approximation, the equation indicates that to secure sharp images emulsions with both high contrast and low turbidity should be selected, also those whose characteristic curves have a small "toe." Various secondary factors which affect sharpness are discussed. Tugman's method of measurement was improved and extended to densities of about 3.

Resolving power of a photographic plate is shown to vary with the sharpness S and with the grain size d according to the *theoretical formula:* $R = 250\, S/(a + bd^2)^{\frac{1}{2}}$. As an illustration, physical development of a Seed 23 plate gives images with one-fifth the sharpness obtained by chemical development, yet the grains were so much smaller that the resolving power was 20 per cent more. The theory also explains observed *variations with time of development and with wave-length, the effect of intensification,* etc. By taking pains to secure fine grains, the resolving power of a wet collodion plate may be made almost equal to the theoretical maximum for the lens used. It is suggested that astronomers are inclined to overestimate the advantage of small grain-size; for the true resolving power of some coarse-grained emulsions is far ahead of that obtainable with large telescopes. As a rule, what is needed is rather higher speed than better resolving power.

[1] Communication No. 90 from the Research Laboratory of the Eastman Kodak Company.

Before developing the general subject of photographic sharpness and resolving power, it will be necessary to define a number of terms current in the literature of scientific photography and to state the most important of the laws governing photographic action.

Fundamental in the theory of photography is the relation between photographic effect and light-intensity and exposure-time, concerning which only a few essentials will be given here. The photographic effect is usually taken as the blackening of the emulsion and is measured by its transparency T, obtained with a suitable photometer, illumination being by completely diffused light. It is customary to employ instead of T a derived quantity D, called density, related to T as follows:

$$D = \log_{10} O \, , \tag{1}$$

where O is the opacity, defined as the reciprocal of the transmission T. The advantage of employing D is that it is proportional to the mass of developed silver; furthermore, it is proportional to visual sensation, i.e., an arithmetical series of densities appears equally spaced in apparent brightness.

Assuming the validity of the reciprocity law, the stimulus is taken as the product of the light-intensity I and exposure-time t. It is designated briefly as exposure E.

With a given plate or emulsion, a given developer and temperature, and a given quality of illumination, the relation between D and $\log E$ is as shown in Figure 1, which gives the curves for two similar and similarly exposed plates developed three and six minutes respectively, in a pyro-developer. It is seen that the relation is nearly linear over a considerable portion of the curves. The slope of the straight portion is designated as γ and is a measure of the contrast quality of the plate. It is found to increase with time of development in accordance with the relation

$$\gamma = \gamma_\infty (1 - e^{-at}) . \tag{2}$$

a is a parameter measuring the velocity of development; γ_∞ is the maximum value of γ obtained from infinite development. The

curve for infinite development obtained by calculation from equation (2) is shown in the figure.

The curves in Figure 1 are designated "characteristic" curves or "H. and D." curves, named after Hurter and Driffield, who

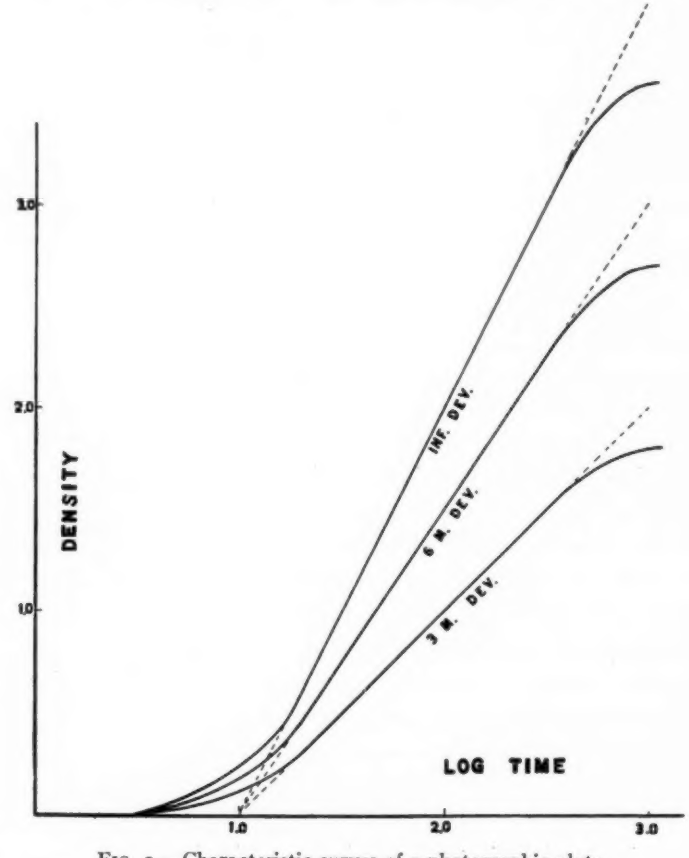


FIG. 1.—Characteristic curves of a photographic plate

developed the system of expressing the relations in this form. In theoretical investigations it is sometimes desirable to be able to write down the equation of such curves. The equation proposed by Hurter and Driffield, namely

$$D=\rho \log_{10}[O-(O-1)e^{-qIt}] \tag{3}$$

containing three parameters, ρ, O, and q, has been found to fit curves actually obtained fairly well. A discussion of this and other formulae will be found in a paper by the writer.[1]

I. GROWTH OF IMAGES WITH EXPOSURE

This subject will be treated first, as it is fundamental and the basis of many of the developments which follow. Any investigation into the photometric properties of the photographic plate depending on size of image, or of the properties of resolving power and sharpness, must start from the assumption of a definite distribution of light at the edge of an image, which diminishes outward in accordance with laws determined by the physical characteristics of the emulsion, such as size, density, and number of grains, their reflecting power, and the opacity of the grains and the gelatine. The color of the incident light is of importance also in determining the distribution. It will be provisionally assumed that the relation between the density of the silver deposit and light-intensity or exposure which holds over "macro"-regions of the plate will hold over "micro"-regions. This provisional assumption is necessary for the mathematical development of the subject. Deviations from this postulate may be classed as secondary phenomena, of which there are many in photography. In fact, the main difficulties of a mathematical treatment of the subject lie in these same secondary phenomena, as will be seen.

Let the relation between the light-intensity I and the distance x from the geometrical image edge be put in the general form

$$\log \frac{I}{I_0} = \phi(x), \tag{4}$$

where I_0 is the light-intensity uniformly distributed over the geometrical image, and ϕ is an unknown functional form to be determined. Let, further, the relation between photographic density D, intensity I, and time t be given by

$$D = F(\log It^p), \tag{5}$$

[1] "On the Relation between Photographic Density, Light-Intensity, and Exposure Time," *Journal of the Optical Society of America*, **4**, September 1920.

where F is a functional form which it is not necessary to specify at the present moment. Schwarzschild's law is assumed to hold. In writing logarithms in (4) and (5) the tacit assumption is made that it is only the sensation-measuring function of I, namely $\log I$, which enters into the problem. Differentiating (5) gives

$$\delta D = F'(\log It^p)\delta \log It^p. \tag{6}$$

It is found experimentally that an image increases in size when I or t increases. In measuring a succession of such images, the edge limiting the image is always taken at a point of constant or fixed density, varying perhaps for different observers. Therefore I_0 and x, or t and x, must vary subject to the condition that $\delta D = 0$. Hence, since F' cannot be zero,

$$\delta D = \delta \log It^p = 0. \tag{7}$$

Assume t constant and I_0 variable. Substitute (4) in (7) and reduce:

$$\frac{d\phi}{dx}\delta x + \delta \log I_0 = 0. \tag{8a}$$

Assume t variable and I_0 constant. There results in this case the similar equation

$$\frac{d\phi}{dx}\delta x + p\cdot\delta \log t = 0. \tag{8b}$$

(8a) and (8b) lead to the following equations for ϕ:

$$\frac{d\phi}{dx} = -\frac{1}{\dfrac{dx}{d\log I_0}}, \tag{9a}$$

$$\frac{d\phi}{dx} = -\frac{p}{\dfrac{dx}{d\log t}}. \tag{9b}$$

It is found experimentally that for a considerable range of exposure the diameter Δ of an image is given by the equations

$$\Delta = a' + 2\kappa' \log I_0, \tag{10a}$$

$$\Delta = a + 2\kappa \log t; \tag{10b}$$

whence, since by definition $\delta\Delta = 2\delta x$,

$$\frac{dx}{d\log I_0} = \kappa' ; \qquad \frac{dx}{d\log t} = \kappa .$$

Substituting these values in (9) gives:

$$\frac{d\phi}{dx} = -\frac{1}{\kappa'} ; \qquad \frac{d\phi}{dx} = -\frac{p}{\kappa} .$$

Integration gives:

$$\phi = -\frac{1}{\kappa'}x + c ; \qquad \phi = -\frac{p}{\kappa}x + c . \tag{11}$$

Substituting the form thus given for ϕ in (4) leads to the required expression for the distribution of light at the edge of an image, as follows:

$$I = I_0 e^{-\frac{1}{\kappa'}x} ; \qquad I = I_0 e^{-\frac{p}{\kappa}x} . \tag{12}$$

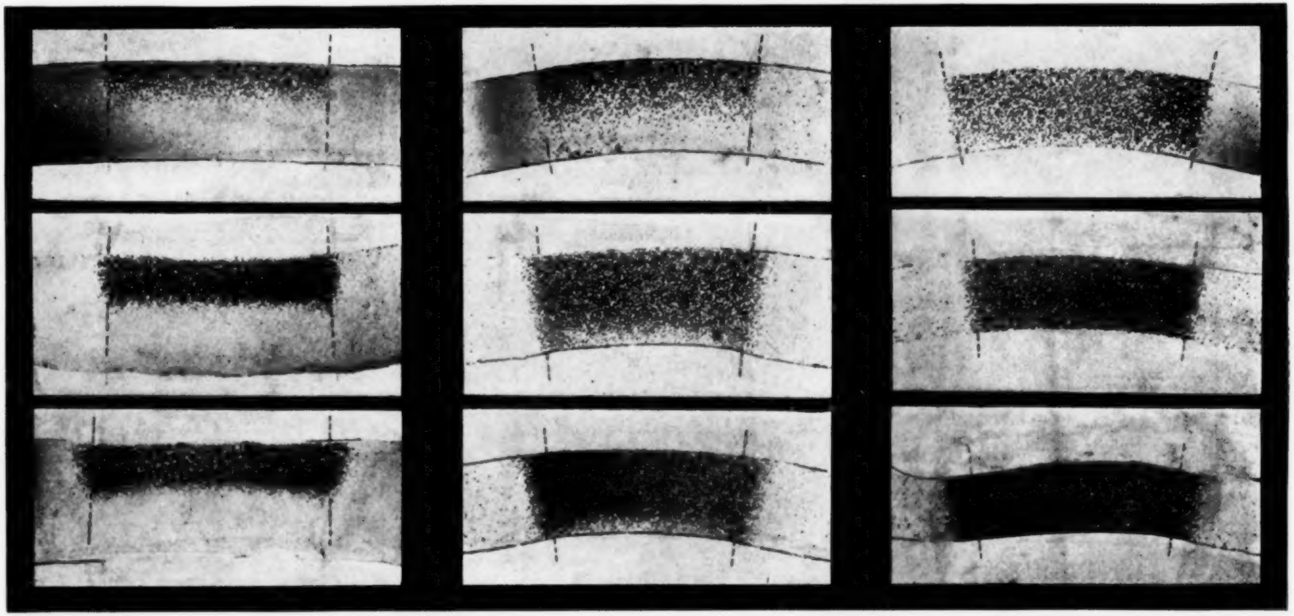
In order that the two values of I shall be identical it is necessary that

$$p = \kappa/\kappa' .$$

Measurements of stellar disks thus afford a means of determining Schwarzschild's coefficient p. Another method of finding p is to determine the ratio of the gammas in the H. and D. curves obtained by varying time and intensity separately. From a brief series of measurements recently made by the writer on artificial stars, a value $p = 1$ was found. Further work is in progress.

It is tacitly assumed in equation (4) that in the region surrounding the edge of the image the light-intensity does not vary downward in the emulsion. The problem has virtually been treated as unidimensional, whereas strictly the problem is a two-dimensional one. The intensity I considered above must be taken to be the average or the mean value of the intensity at the distance x from the edge of the image. That this is a fair approximation to the true condition will be evident from Plate VIII, showing a series of magnified sections of sharp slit-images of various exposures and strength of development. At any given distance from the true edge (dotted line)

PLATE VIII



MAGNIFIED SECTIONS ACROSS SLIT-IMAGES. SECTIONS WET

there does not appear to be any considerable variation of density of deposit with depth. Figure 2 shows, on the other hand, a drawing of sections of equiluminous surfaces *a* to *g* as they are imagined to lie. They will vary in position and shape with the relative value of the turbidity of the emulsion and the adjustment and perfection of the optical system. Starting at the edge of the image the equiluminous surfaces have a comparatively small inclination to the horizontal, which increases on passing outward, until at *g* the surface becomes nearly perpendicular to the surface of the emulsion, thus approximating the conditions laid down in the formula. It is manifestly impossible to treat the problem with

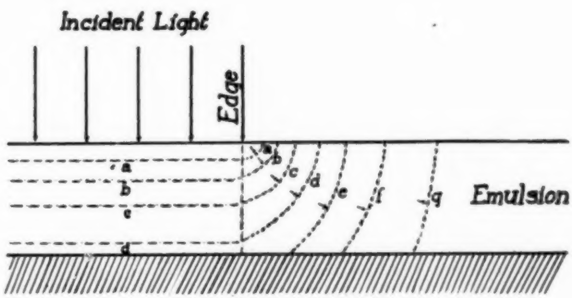


FIG. 2.—Probable form of equiluminous surfaces in an emulsion

mathematical rigor on account of the unknown relative value of the various factors. In correlating this drawing with the actual sections just shown, allowance must be made for the fact that the sections are swelled, the vertical scale being about ten times the horizontal.

The formula which has been derived for the distribution of light outward from the edge of an image

$$I = I_0 e^{-\frac{1}{\kappa'}x}$$

is seen to be nothing more nor less than Beer's law for light-intensity within a homogeneous medium, with extinction coefficient $\frac{1}{\kappa'}$. If, however, the light-distribution in the film in a direction

perpendicular to this, i.e., perpendicular to the surface of the film within an image and some distance from any edge is considered, it is quite likely that a different law will be found to hold. This is due to the fact that the diffuse and specular transmissions of an emulsion are very different, the diffuse transmission being higher, so that the upper layers of an emulsion illuminated by specular or nearly specular light transmit proportionally less than the lower layers which are illuminated from the layers above by completely diffuse light. The same reasoning can be applied to the distribution outward from the edge of an image, so that Beer's law should be found not to hold. However, in this case there is the modifying influence of effects of diffraction and aberration which equally with the phenomena of turbidity are potent in forming the resultant curve of light-distribution. In other words, the flux at any point in the emulsion outside the true image is a combination of flux from the image transmitted through the emulsion with the flux from the surface depending on the optical system. Apparently combination of the two effects leads to the very simple distribution of Beer's law. But Beer's law is only an approximation, as will now be shown. Figure 3 gives a series of diameters of star images, plotted against logarithm of the exposure-time as abscissa. The images are of an artificial star made with a lens of

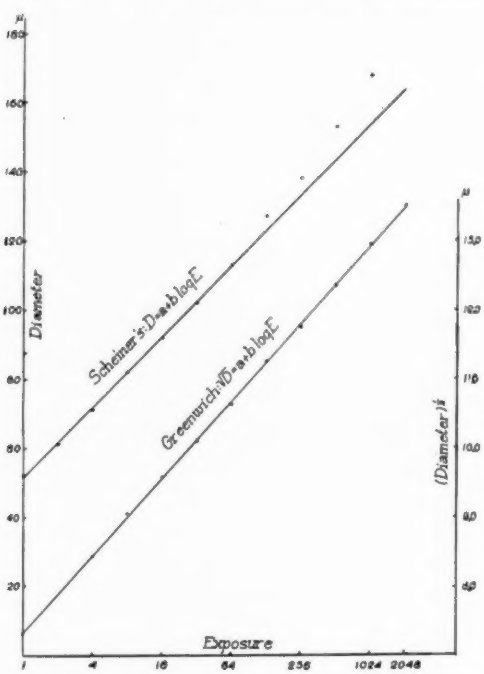


FIG. 3.—Scheiner's and Greenwich formulae compared with measures.

6 inches' focal length, the focus being very accurately adjusted. It is seen that the straight-line relation between Δ and log t,

$$\Delta = a + b \log t \text{ (Scheiner's formula)}, \tag{13}$$

which has been found above by mathematical reasoning to be true provided the distribution of light is according to Beer's law, holds only for a limited range of exposure. The same series of diameters has been plotted on the diagram according to the Greenwich equation

$$\Delta^{\frac{1}{2}} = a' + b' \log t. \tag{14}$$

It is seen that in this case the straight-line representation is nearly perfect. The following conclusion can be drawn:

Making allowance for continuation of the curves backward to threshold diameter, for which $\Delta = 0.022$ mm approximately, it appears that the Scheiner formula holds over a range of exposure-times of approximately 1 to 500. On the other hand, the Greenwich formula is seen to hold over a range of approximately 1 to 15,000, or 30 times greater. For most purposes, however, the Scheiner or, as it is otherwise called, the Charlier formula, is sufficiently accurate, so that Beer's law can be applied to the light-distribution outward from the edge of an image without serious error. This is of great advantage as it allows of the further theoretical development of the subject of sharpness of image and of resolving power along comparatively simple lines.

Referring to Figure 1 it is seen that over a considerable range of exposure the following linear relation holds:

$$D = a + \gamma \log E. \tag{15}$$

This is seen to be of the same form as (13). From analogy the writer proposes the term "astrogamma" (Γ) for the coefficient b in equation (13). Expressed in words, astrogamma is the log exposure rate of increase in diameter of stellar images, just as in photographic literature γ is the log exposure rate of increase of density of silver deposit. If the unit of exposure is taken as one stellar magnitude we should have

$$\text{astrogamma } (\Gamma) = 2.5b,$$

so that in this case astrogamma would be defined as the increase in diameter of a stellar image for unit increase in stellar magnitude.

Figure 4 shows a series of three curves of light-distribution at the edge of sharp slit-images which were obtained by contact printing. The slit is formed of two inclined steel knife-edges very accurately ground and adjusted. The necessary factors $\frac{1}{\kappa}$ were obtained by making use of equation (10); p is assumed equal to unity. Monochromatic light of wave-lengths 4000, 4800, and

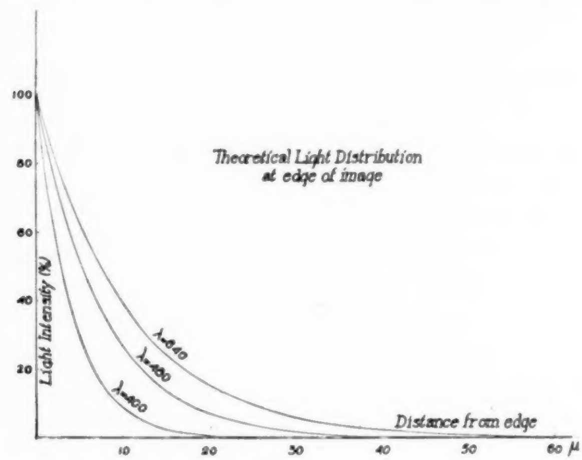


FIG. 4

6400 was employed. The diminution of visual sharpness with increasing wave-length is clearly shown. At 10 μ from the edge the light-intensity for red light has been reduced to 0.39 I_0, for blue light to 0.26 I_0, and for violet light to 0.09 I_0. The effect of wave-length on photographic sharpness will be shown in Section 2.

2. SHARPNESS OF THE PHOTOGRAPHIC IMAGE

The sharpness of a photographic image is definable by the equation

$$S=\frac{dD}{dx}, \tag{16}$$

where D = density and x is measured from the edge as origin in a direction perpendicular to the edge. Since visual sensation is

proportional to D, the rate of change of D as the edge of the image is passed over is clearly a logical mathematical definition of the subjective phenomenon of sharpness or, in other words, is an expression for the sensation of sharpness.

The developments in the preceding paragraphs have shown that D is a known function of light-intensity (3) and that light-intensity can be expressed as a function of the distance from the true edge (12). I_0 being the only intermediary variable, we accordingly have

$$S=\frac{dD}{dx}=\frac{dD}{d\log I}\cdot\frac{d\log I}{dx}.$$

It is convenient to express I in common logarithms, as follows:

$$I=I_0 10^{-\frac{1}{\kappa'}x} \text{ or } I=I_0 10^{-\frac{p}{\kappa}x};$$

which gives on differentiation

$$\frac{d\log I}{dx}=-\frac{p}{\kappa};$$

also

$$\frac{dD}{d\log_{10} I}=\frac{\gamma}{p},$$

where γ is the contrast factor or gamma, assumed to be obtained by increasing exposure-time. The maximum sharpness is then

$$S_m=-\frac{1}{\kappa}\gamma. \tag{17}$$

It is instructive to look at the matter from a different point of view. The characteristic curve of an emulsion is obtained by making a change of variable in (3) as follows:

$$\xi=\log It, \text{ or } \log I=-\ \log t+\xi, \tag{18}$$

ξ being the new abscissa. But the transformation made to obtain the curve of sharpness is of exactly the same form, namely,

$$I=I_0 10^{-\frac{p}{\kappa}x}, \text{ or } \log I=\log I_0-\frac{p}{\kappa}x. \tag{19}$$

Comparing (18) and (19) it is seen that the characteristic curve of the emulsion, the $D:\xi$ curve, is exactly the same as the curve of sharpness, the $D:x$ curve, the only difference being that the abscissae have been compressed by the factor $\frac{p}{\kappa}$. The curve is also reversed, through the negative sign in (19). This leads to a simple rule for determining the sharpness in any case. Mark on the characteristic curve a point where the density is equal to the density of the image whose sharpness is to be determined. Find the maximum slope of the curve below this point, γ_m. The required sharpness is then

$$S=\frac{\gamma_m}{p}\frac{p}{\kappa}=\frac{\gamma_m}{\kappa}.$$

For low densities, on the "toe" of the characteristic curve, it will be necessary to replace γ by the average value of the slope obtaining below the point considered. As a further corollary, to obtain the complete curve of sharpness for any exposure, mark off the given exposure on the X-axis and draw the ordinate. That portion of the characteristic curve lying within this ordinate, compressed horizontally by the factor $\frac{1}{\kappa}$, will be the required curve of sharpness. It will be instructive to study a practical example.

A series of contact prints of the narrow slit already described were made on a photographic plate, using light of wave-length 4800 A, each exposure-time being double the preceding. The plate was developed to a γ of 1.65. The rate of growth of the slit was measured, from which the distribution of light was obtained in accordance with the foregoing theory, as follows:

$$I=I_0 10^{-.058px}.$$

It is of interest to note that according to this formula, assuming $p=1$, the light-intensity at a distance of 5 μ or 2 grain-diameters from the geometrical edge is reduced to 51 per cent; at 17 μ, it is 10 per cent; at 35 μ, it is 1 per cent.

It is necessary to give a numerical value to the parameter O in (3). This was taken as 100, a value which fits the average emulsion. For $\gamma=1.65$, it is found that $p=1.90$. Defining $Q=qIt$

the value Q_0 of Q for the lightest exposure was chosen $Q_0 = 0.001$. The values for the succeeding curves were $4Q_0$, $16Q_0$, $64Q_0$, ... $264144Q_0$. The computed curves of density-gradient and their relation to the geometrical edge are shown in Figure 5. It is seen that for light exposures the density-gradient or sharpness is low, and that even for the useful density of unity, the sharpness is only

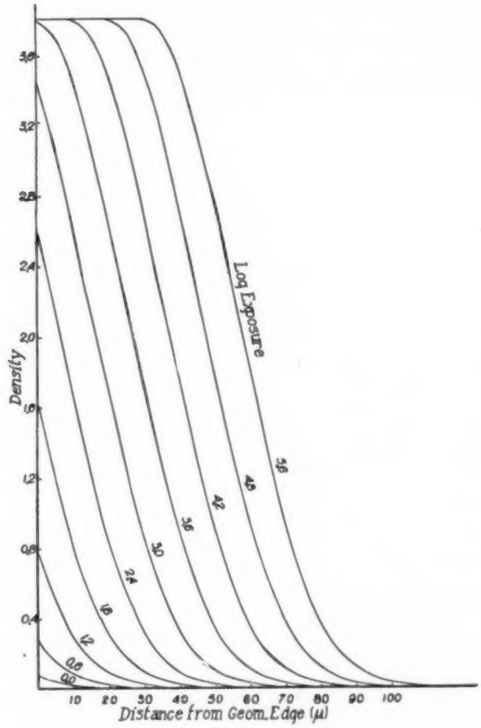


FIG. 5.—Theoretical sharpness curves

70 per cent of the maximum. The formula for sharpness (15) which is usually given is therefore true only for high densities, if (3) is correct, and is seen to be considerably in error in the useful cases of moderate densities of the image, which are the rule in cases of high resolving power. But in the case of most emulsions the slope of the characteristic curve is greater for low exposure than the H. and D. equation (3) indicates. This means that the formula

for sharpness $S=-\frac{1}{\kappa}\gamma$ is correct over a wider range of exposures than the formula gives, which is borne out by measurements of sharpness.

The foregoing developments indicate the type of emulsion which should be used to secure sharp images. First, it must be one of high γ and low turbidity κ. Second, it should belong to type A, Figure 6, having a minimum amount of "toe." For in cases that arise in practice the density of images is low, generally less than unity, so that it is very important that a maximum slope

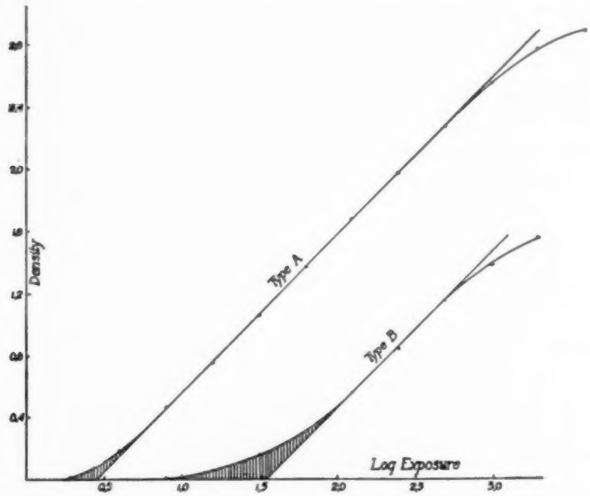


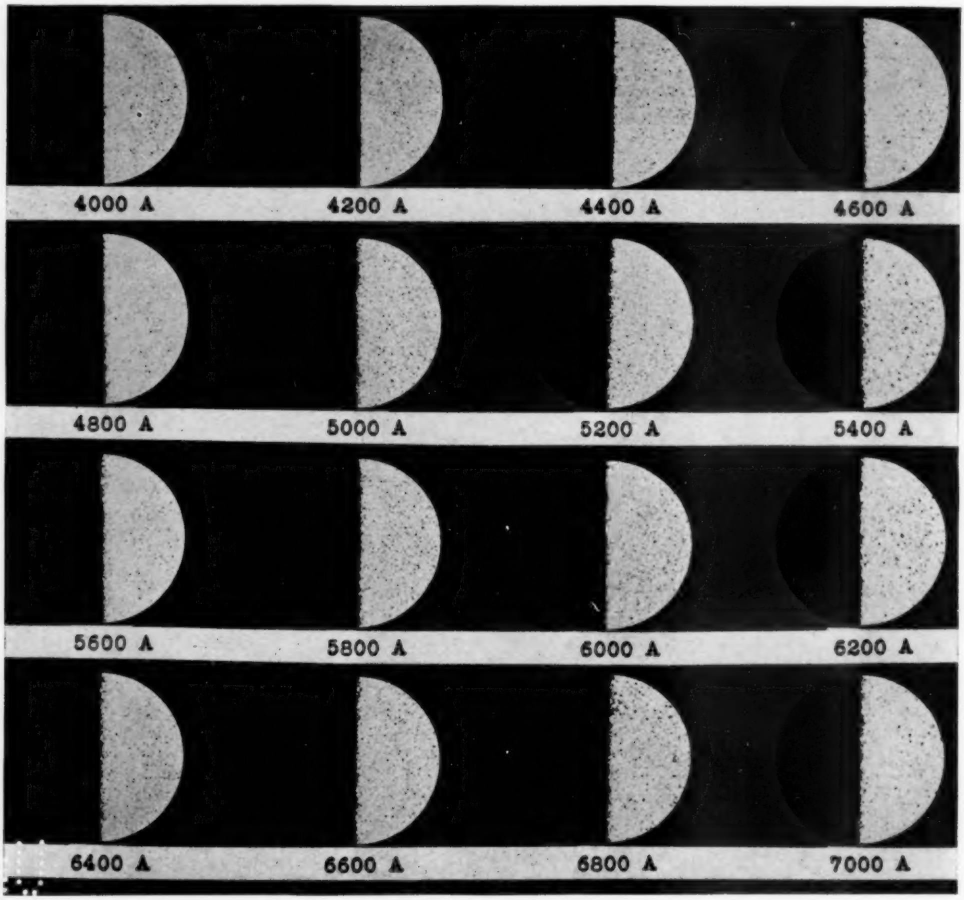
FIG. 6.—H. and D. curves for two emulsion types

of the characteristic curve should be reached at a comparatively low density. For this reason emulsions of type B, Figure 6, are quite unsuited to give sharp images.

Measurement of sharpness.—So far as the writer is aware the only measurements of sharpness of image so far made are by O. Tugman.[1] He used two distinct methods: (*a*) a Koch registering microphotometer, and (*b*) projecting a highly magnified image of the edge to be measured upon the nose of a König-Martens photometer over which narrow slits are placed. The basis of the two

[1] *Astrophysical Journal*, 42, 331, 1915.

PLATE IX



Enlargements of Knife-Edge Contact Prints on a Panchromatic Plate Exposed to Light of Varying Wave-Length

methods is the same, a microscope being employed in each case to produce a magnified image. In collaboration with Dr. P. G. Nutting, the writer has developed and perfected the second method used by Tugman. Largely because of scattered light, the methods used by Tugman were incapable of measuring densities greater than about 1.2, all higher densities registering the same value. This lack of sensitivity is a serious drawback to accurate measurement of density-gradient. The improvements designed were in the direction of increasing the sensitiveness. Success was attained to such a degree that it is now possible to measure with the perfected set-up densities up to about 3.0. This was attained by using an arc as the source of illumination; by replacing the high-power microscope-objective having many air-glass surfaces by one of lower power with few free surfaces; by removing the eyepiece of the microscope, obtaining magnification by an increased "throw"; and lastly by compressing the magnified image in the direction of its length by means of a cylindrical lens. The instrument will be more fully described in another paper. On account of the effects of residual scattered light, and the fact that illumination is by specular light, it is necessary to calibrate the instrument to read diffuse densities.[1] This is easily done by comparing a series of densities measured on a König-Martens photometer, using completely diffused illumination, with the same series measured on the microphotometer.

It is only possible to touch briefly upon the many problems arising in the use of the microphotometer, the results so far obtained and their interpretation. Perhaps the most important of the many series of measurements so far attempted has been the determination of the sharpness of images made with light of different colors, or, in other words, determination of the curve of sharpness-variation with wave-length. Plate IX shows a series of edges printed by contact in the manner already described, using panchromatic emulsion, exposure being to light of wave-length 4000 A to 7000 A. The variation of sharpness is strikingly apparent. The original

[1] There is a difference in measured densities, depending upon the character of the illumination, whether by specular, by diffuse light, or by a combination of the two. The ratio of specular to diffuse density, which is greater than unity, reaches in some cases a value of 1.5. It is generally called q.

images, of which there were ten at each wave-length, were measured on the microphotometer. The resulting sharpness, expressed as the fall of density (diffuse) in a distance of one micron, is shown in Figure 7.

At this point the opportunity presents itself to "check up" on the theory developed in the preceding paragraphs. For it is by

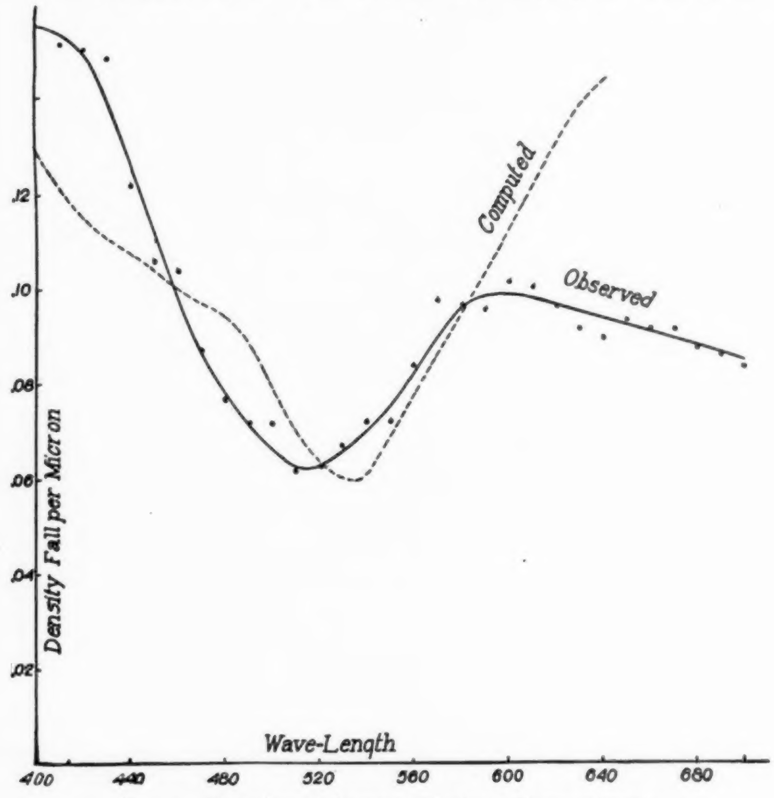


FIG. 7.—Variation of sharpness of edge with wave-length

no means certain that in a region only a few grains wide the photographic action will follow and record the extreme range of light-intensity which exists in this limited space. We know from other considerations that it will not, at least that there are secondary effects which modify the main phenomenon. But the developments of the present paper are concerned principally with primary

actions, with deriving the laws of the fundamental or basic phenomena. The continued and complete development of the subject must deal with secondary phenomena as well; indeed sufficient mention will be made of them in what follows to show that there is a wide field for their study. It was proved above that for high

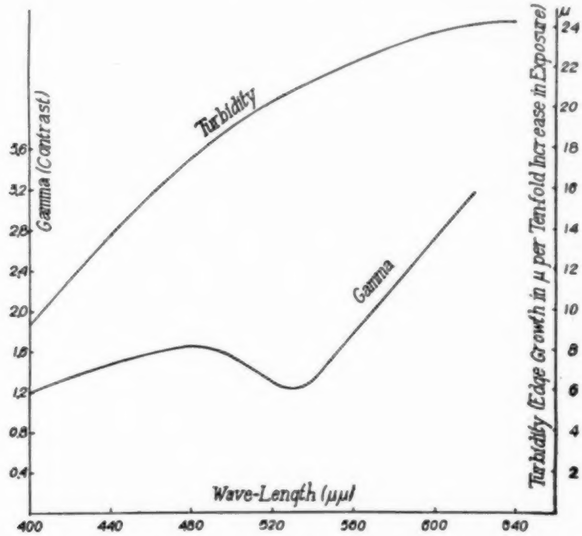


FIG. 8.—Variation of turbidity and contrast with wave-length

densities, such as occur in the contact images whose edges have just been measured, the sharpness is given by

$$S = -\frac{1}{\kappa}\gamma.$$

γ can be measured in the ordinary way. It was shown above how to measure κ. Both are found to be functions of wave-length, as will be seen from Figure 8, in which the results of measurement are plotted. Combining the two curves according to the foregoing formula gives what can be called a theoretical sharpness-curve. This has been plotted on the same diagram with the measured sharpness-curve (Fig. 7). The agreement of the two curves is sufficiently close for the conclusion to be drawn that the primary action is as imagined and lends itself to a simple mathematical

treatment. The wide difference between the two curves in the red is especially marked. This is a point which requires clearing up, for which more data will be needed. Determination of the contrast or γ in the red end of the spectrum appears for some unknown reason to be subject to very great uncertainty, variations of as much as 50 per cent being found in the same emulsion for definite wave-lengths in the red. It is quite probable that the disagreement of the two sharpness-curves in the red portion of the spectrum is related to this uncertainty or instability in contrast. Disagreement in the violet end on the other hand is probably due to other causes, of which the chief is "reciprocity" failure.

The first extensive investigation undertaken with the microphotometer was the measurement of the sharpness of knife-edge images, exposed to λ 4400, the plates being developed in many different kinds of developers for various times. The work as a whole is to be repeated as a check before detailed results are given. Laboratory tests on resolving power showed a variation of resolution depending on developer and time of development. It was considered that a parallel variation should occur in sharpness of image. The relation of resolving power to sharpness of image will be taken up in the next section. Equation 17 shows that sharpness S depends upon developer only to the extent that γ or contrast depends upon the developer. From measurements so far made, however, it appears that this is not the whole story, that there are secondary effects entering which modify the main phenomenon. For example, while it has been found that, for the same developer, S increases with the degree of development, γ, at least for heavy densities, the relation is not linear as it should be according to (17). If upon repetition this should be verified it would be but another case of the action of secondary phenomena, the explanation of which is to be sought in the byways of photographic theory.

It will be useful from many points of view to give the details of one series of measurements of sharpness of image. The degree of accuracy to be expected in work of this kind will thus be made apparent. A series of knife-edge contact prints on a standard emulsion were made with exposure to light of wave-length 4200 A. Ten exposures were made on each of three strips. The strips were

developed for three-fourths minute, one and one-half minutes, and three minutes, respectively, in caustic hydroquinone. The exposure-time for the images on each strip was so chosen that the density developed to approximately unity in all cases. The edge-gradient of each image was measured with the microphotometer at a magnification of 208 (the slits over the nose of the König-Martens photometer are 0.55 mm wide). Each edge was measured twice, at different portions of the edge. There are thus twenty curves of edge-gradient for each development time. The individual values are given in Table I.

TABLE I

MEASURED EDGE-GRADIENTS S

Image	Three-fourths Minute's Development $\gamma=0.71$		One and One-half Minutes' Development $\gamma=1.16$		Three Minutes' Development $\gamma=1.48$	
	First Measurement	Repeat	First Measurement	Repeat	First Measurement	Repeat
1	3.23	2.78	2.63	3.03	2.23	3.03
2	2.33	2.78	2.22	2.38	3.23	2.94
3	3.33	2.78	2.86	3.03	3.12	2.94
4	3.23	3.03	2.38	2.22	2.38	2.78
5	4.00	3.12	3.33	2.22	2.63	2.63
6	3.23	2.63	2.86	2.78	3.12	3.92
7	2.70	2.50	2.12	3.33	2.17	3.12
8	2.61	3.70	2.70	3.57	3.33	2.78
9	2.78	2.86	2.22	2.63	3.03	2.56
10	2.94	2.63	3.23	3.03	3.12	2.94
Means	2.96 ± .09 (p.e.)		2.73 ± .10		2.90 ± .09	

To reduce S in this table to absolute values (fall of density per micron), the numbers are to be multiplied by 0.0416. By comparing the individual values of S with the mean, the average error of a single determination of the edge-gradient is found to be ± 0.33 or 12 per cent. For photographic work this is not excessive. Repeats on the same section of edge show differences of only 3 per cent, indicating that the greater part of the error is photographic, that is, is due to photographic variations. The most difficult case has been chosen for measurement, namely, that of an exceedingly sharp image of low density. For the average edge, where the total

fall of density is less abrupt, more accurate and consistent values are obtainable.

In order to compare measured density-gradients or sharpness with theory, the turbidity factor κ and contrast factor γ are necessary. κ is determined from the rate of growth of the slit images as already explained; γ from a series of exposures in a monochromatic sensitometer (log t variable). In this case it was found that the slit images increased $5.0\ \mu$ (0.0050 mm) per doubling of exposure-time. This gives for κ:

$$\kappa = \frac{1}{2 \log 2} \times 5.0 = 8.3\ \mu.$$

It will be recalled that the theoretical sharpness is obtained by multiplying $\frac{1}{\kappa}$ by the maximum value of the slope of the characteristic curve of the emulsion which occurs below the density of the image in question. The density unity of the slit images is at least on the straight portion of the curve, so that the simple formula applies,

$$S = \frac{1}{\kappa}\gamma = 0.120\,\gamma.$$

The values of S computed from this formula are compared with the values obtained by direct measurement in Table II.

TABLE II

	γ	Observed S	Computed S
Three-fourths minute's development	0.71	0.123	0.085
One and one-half minutes' development	1.16	.114	.139
Three minutes' development	1.48	.121	.178

The computed values are seen to lie on both sides of the observed values. The observed values are nearly constant; the computed values of course vary directly as γ. The explanation of the discrepancy must be again sought in secondary phenomena. Agreement is such that there is no doubt of the primary action being as postulated. For γ unity or for normal development

agreement between theory and observation is seen to be complete. For lower development there is a secondary factor tending to increase sharpness; for higher development a factor tending to decrease it. These will be considered in turn.

In the case of weak development the silver grain forming the image is in an incompleted state. This is seen very clearly in a microscopic study. The image as a whole looks grayish, compared with the dense black obtained with higher development. The measured densities of the two images compared are equal in the cases considered. Under the microscope, assuming short development, a large number of shadowy, indistinct grains can be seen which cannot be focused. They are due to partial development. Grains are known to develop in spots, with numerous centers of development in each grain. This process being arrested in short development, the result is a grain composed of gelatine imbedding scattering particles of metallic silver. The grain therefore transmits and scatters the viewing light, causing both grayish and shadowy appearances, depending on its degree of development. It is also seen that small grains are relatively numerous for short development. This is as we should expect, on account of the time factor in completing development of a grain. It is clear that short development must give a plate of relatively fine grain. But the size of grain does not enter into the formula for sharpness, a statement which seems contrary to accepted views and to the measurements of sharpness which are now being discussed.

There is other evidence indicating that silver bromide grains receiving a heavy exposure E are more developable as well as more quickly developable than grains receiving less exposure. Consider the case of a sharp image formed from a weak exposure and receiving short development, as in the case under examination. The latent image in the region bordering the true edge must be in this case relatively weak and is accordingly not developed when the time of development is short. Sharpness must therefore be increased. Longer or normal development ($\gamma = 1$), on the contrary, brings out the true latent image. In this case the formula for sharpness is found to hold. This explanation finds support when the case of very short time of development is examined. For example, when

images on a plate having received a development of only twenty seconds are measured, it is found that the sharpness has decreased considerably. This shows that the diminishing effect of a decreasing γ is becoming effective, more than counterbalancing the increasing effect due to non-developability of the latent image.

It is a common impression that small grain-size is the cause of sharpness of image. It is true that in the case of resolving power, as will be shown in the next section, size of grain is an important factor. But the matter of image sharpness is quite different. This depends primarily as shown upon contrast and turbidity. Usually fine-grain plates have high contrast (γ) and low turbidity (κ), which accounts for the grain being undeservedly given the credit for the sharpness or lack of it. Sharpness due to uniform random distribution of silver grains, a popular conception of the subject, is of a higher order than occurs in real emulsions.

The decrease in the theoretical sharpness for long development remains to be explained. The cause is to be found in the phenomenon of "graininess" or irregular aggregating of the silver grains, which appears to increase with development. While of great importance, its discussion is beyond the scope of the present paper.

Some cases of secondary phenomena taking place at the edge of an image are worthy of brief mention. The phenomenon of an apparently luminous band at the border of a dense image commonly seen in moving pictures is one of these. Again, Plate VIII, showing sections of sharp slit-images, reveals an added mass of silver grains in the lower layers of the emulsion, immediately under the edge itself. These are both effects of developer, due to a combination of bromiding and exhaustion of the developer with clogging of the channels in the gelatine possibly. They are important and interesting phenomena, but cannot be considered at length in this place. Undoubtedly these phenomena are related to the Eberhard and Kostinsky effects, which will be treated in a following paper.

Another secondary effect of a totally different class arises in the measurement of densities near the edge of a sharp image. It would seem that a given mass of silver at the edge of a large image should measure photometrically higher than when placed in the center of the image. The amount of light transmitted by any

given small region depends upon its direct illumination and upon the amount of light diffused and reflected into it by the surrounding regions. At the edge diffusion is entirely lost on one side, so that less total light is transmitted, leading to a higher measured density.

3. PHOTOGRAPHIC RESOLVING POWER

So far as the writer is aware, the only formula which has been proposed for photographic resolving power is by Wadsworth.[1] His result, namely, that the linear resolving power is equal to four times the diameter of the developed silver grain, was arrived at by some photographic considerations combined with the usual diffraction theory of visual resolving power. It is well known that the resolving power is considerably less than the value given by the Wadsworth formula.

Study of the problem has led to a separation of the factors controlling resolving power into two, namely, (*a*) sharpness factor, (*b*) grain factor.

Consider a pair of parallel line images aa and $a'a'$ on the photographic plate, of equal width, separated by a space w equal to the common width of the lines. Defining resolving power as the number of lines per millimeter just separated, we have

$$R=\frac{1000}{2w}\ (w \text{ in microns}). \tag{20}$$

It is not possible to derive any formula for R depending on physical data alone. It can be postulated that for resolution a certain contrast must exist between the brightness of the images aa and the center of the separating space b before resolution takes place. But brightness as a sensation is measured by density of the photographic deposit, so that it can be said that for resolution there must be a certain difference in density, which may be called Δ, between the images aa, $a'a'$, and the center of the separating space b. It is quite evident that Δ is a function of the diameter of the grain. For it is easily seen that if the grain of the plate is very fine, a relatively slight difference in contrast Δ between a and b will be sufficient for resolution. With increased size of grain, the difference

[1] *Astrophysical Journal*, **3**, 188, 1896.

of contrast must increase. The relation between size of grain and minimum contrast is to be regarded as belonging to the class of physico-psychological laws of vision, obtainable as empirical relations by direct measurement, which are met at every turn in physical investigations in which vision plays a part. It is perhaps needless to say that no direct data are available on the subject. From general considerations it is to be expected that the difference in contrast will approach a minimum value semi-asymptotically as the size of grain approaches zero. The simplest curve fulfilling the properties as imagined is probably the hyperbola, with two arbitrary constants. Accordingly

$$\Delta=(a+bd^2)^{\frac{1}{2}}, \tag{21}$$

where d is diameter of grain, and a, b are arbitrary constants to be determined by experiment. It should be noted that Δ is to a certain extent a function of the intensity of the viewing illumination, a complication not necessary to consider here.

Let it be assumed that the fall of density from the edge of the image a to the center of the separating space b is uniform. Let S be the sharpness of the images aa or the rate of fall of density, as defined in Section 2. Let it be further assumed that the diffraction and aberration patterns of the two lines aa and $a'a'$ are separated. This will be approximately true if the optical system is a good one. With these assumptions it is not difficult to derive a formula for resolving power; for by definition

$$\Delta=\frac{w}{2}S, \text{ or } w=2\frac{\Delta}{S}.$$

Substituting for Δ and w their values in terms of size of grain and resolving power,

$$R=\frac{1000\,S}{4}(a+bd^2)^{-\frac{1}{2}}; \tag{22}$$

or since $S=\frac{1}{\kappa}\gamma$,

$$R=250\frac{\gamma}{\kappa}(a+bd^2)^{-\frac{1}{2}}. \tag{23}$$

It should be possible by a series of suitably planned experiments to determine the parameters a and b in this equation. This is a

subject for future investigation. It will be of interest to determine Δ in a particular case for which data are available. It was found, for example, that for a certain emulsion the resolving power was 70, determined from artificial double stars at wave-length 4500 A. The turbidity κ was found by measuring the time-rate of increase of the diameter of artificial stars. This was found to be 9 μ for each doubling of exposure-time, whence $\kappa = \frac{9}{2 \log 2} = 15.0$. γ can be taken equal to unity. For the determination of Δ therefore

$$R = 250 \frac{\gamma}{\kappa} \Delta^{-1} ,$$

giving

$$\Delta = 0.24.$$

That is, for an emulsion of medium size of grain ($d = 2\ \mu$ to $3\ \mu$) the contrast necessary for resolution measured in density is equal to 0.24, or in light-intensities, since $D = \log \frac{T_1}{T_2}$, a ratio of 1 to 1.75 is necessary.

Experimental results on resolving power have been studied with a view to the general application of the formula for resolving power deduced above. They will be briefly considered.

It is well known that fine-grain emulsions give higher resolution than those of coarse grain. A particularly interesting case is the relative resolving power of an average emulsion, such as Seed 23, for chemical and physical development respectively. Obviously the turbidity factor κ is the same in the two cases. It is quite different, however, with the contrast factor γ. For physical development γ was found to be 0.5, for chemical development, 2.5. Since sharpness $S = \frac{\gamma}{\kappa}$, it would be expected that the sharpness under physical development would be very much less (five times) than under chemical development. This is actually found to be the case, the difference being strikingly evident to the eye, without measurement. Since the sharpness of outline is so much less for physical development, it might be expected that the resolving power would likewise be less. As a matter of fact it was found that the resolving

power under physical development is actually higher by 20 per cent. We have here an interesting case of the balancing of the factor of sharpness against that of the size of the grain. In physical development the grains are exceedingly small, many times smaller than under chemical development. The drop in resolving power due to a low factor of contrast is more than made good by the accompanying decrease in size of grain.

A further testing of the formula is to be found in a study of resolving power depending on developer and time of development. A large amount of data on this subject has been obtained by K. Huse.[1] He finds that the resolving power with increasing time of development rises quickly to a maximum, remains constant for a considerable interval, then drops. Here again it is a question of changes in contrast and the size of the grain opposing each other. Variations with developer are to be explained similarly.

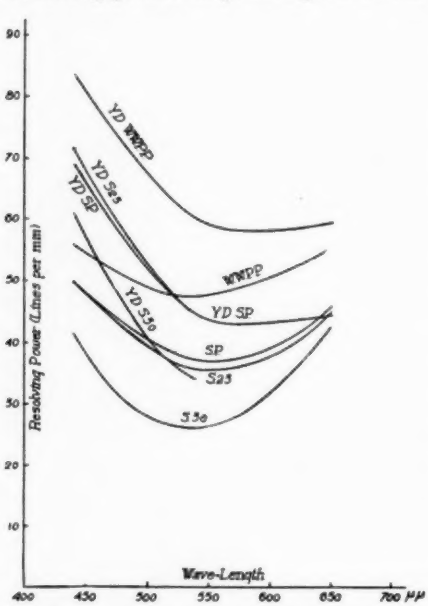


FIG. 9.—Variations of resolving power with wave-length for various plates.

A particularly interesting case is the variation of resolving power with color. It is found that the resolving power is high in the violet and blue, diminishes to a minimum in the green, rising again toward the red. This phenomenon can be explained by referring to the curve of wave-length and turbidity, and the curve of wave-length and γ. In the violet the turbidity factor is high and γ low, while in the red the turbidity is low and γ high. Combination of the two curves is found to give a minimum in the green, agreeing with observation. Figure 9 shows the observed variation of

[1] *Journal of the Optical Society of America*, **1**, 119, 1917.

resolving power with wave-length for several classes of emulsions. The fan test-object devised by Dr. C. E. K. Mees was used in making these measurements. Monochromatic filters were employed. To give some idea of the delicacy of adjustment necessary in work of this kind, it was found that working at $F/8$ an error of focus of 0.03 mm can be detected by its effect on the resolving power. The camera therefore must be focused for each color separately. The color-curve of the objective thus found agreed exactly with that found on a precision lens-bench. The objective is a Fuess single-achromat of 6-inch focus working up to its full visual resolving power. Resolving power determined by means of the fan test-object does not agree with values determined from parallel-line or double-star test-objects. The latter two give values 40 per cent higher on the average. The subjective factor already considered is apparently the cause of this difference, Δ being greater for the fan test-object.

A few other cases will be only briefly touched upon. As is well known, bathing plates in yellow dye increases the resolving power, the increase being more than 50 per cent. Measurements of turbidity and γ show the cause of the increase. It was found that the turbidity factor $\frac{1}{\kappa}$ increases considerably after bathing. Some portion of this increase, however, is nullified by a decrease in γ amounting to 20 per cent. The product of the two, upon which resolving power partly depends, is on the whole increased by an amount sufficient to explain the observed increase. It should be mentioned that the speed of a plate is diminished tenfold by yellow dyeing.

A case of resolving power showing the probable influence of the failure of "reciprocity" was observed. It was found that for a weak light-source, other factors being the same, resolving power was increased about 20 per cent compared with that for a strong source. In this case the light scattered in the spaces between the lines is so exceedingly low in intensity as to be below the threshold, so to speak, of the plate, thus diminishing the effective turbidity. There may also be other factors entering, such as variation in the size of grain.

An interesting case is that of the resolving power of the wet collodion plate. It is well known that the size of the grain of the collodion plate is subject to control in the development operation. By using nearly monochromatic violet light and by exceedingly careful focusing, in conjunction with the proper conditions of development, it was found possible to obtain on this plate a resolving power of 175, or nearly equal to the theoretical resolving power of the lens used. This is approximately equal to the resolving power obtained from albumen plates.

It was found that intensification does not increase resolving power. It is useful, however, for it brings out the resolution more clearly when actually present.

Other secondary factors affecting resolution are (*a*) image contraction, which tends to increase resolving power; (*b*) an effect pointed out by Goldberg[1] due to the difference in the sensitivity of the grains composing a high-speed emulsion. Briefly stated, he considers that an irregularity of outline of an image leading to diminished resolution may be produced by numbers of highly sensitive silver bromide grains lying just without the edge of the image which are infected and made developable by the comparatively weak light to which they are exposed, combined with numbers of insensitive grains lying just within the edge of the image too insensitive to be infected even by the strong light to which they are exposed. It will be noted that both of these effects if operative modify the sharpness of image as well as resolving power, so that equation (22) for resolving power holds if the observed S and not the theoretical sharpness $\frac{\gamma}{\kappa}$ be used.

In applying the results obtained for sharpness and resolving power to a particular problem caution must be used. From the fact that increased sharpness and resolving power have been secured by using violet light, and again by using yellow-dyed plates, in the particular experiments described above, it must not be assumed that the results are of general applicability. For, as shown, sharpness and resolving power are directly proportional to gamma, and inversely proportional to turbidity. Turbidity is decomposable

[1] *Photographic Journal*, **36**, 315, 1912.

into two parts: (1) the inherent turbidity of the emulsion; (2) the turbidity arising from the optical system. The results obtained in this paper were secured by making the turbidity due to the optical system zero or nearly so. Under average conditions of astronomical photography, due largely to atmospheric dispersion and unsteadiness, the optical turbidity becomes very large, completely overshadowing the inherent turbidity of the emulsion. In this case then, the gamma, or contrast property of the photographic plate, becomes the determining factor. Plates of high contrast should accordingly be chosen to secure maximum sharpness. Again, since it has been shown above that the contrast is low in the violet, and increases with increasing wave-length, we conclude that isochromatic plates used with a yellow filter should give greater sharpness and resolving power in ordinary astronomical work, as is actually found to be the case. Still better results should be secured by photographing in the red. In all cases of this kind where the optical turbidity predominates, the gamma wave-length curve of Figure 8 becomes the true sharpness wave-length curve in place of the curve shown in Figure 7, which holds only for an ideal or practically perfect optical system.

The reason why yellow-dyed plates are not suitable for astronomical work is now clear. Such plates owe their high resolving power to a low inherent turbidity which, however, as just shown, is of no help in the cases arising in astronomy. But, as shown above, yellow-dyed plates have a low contrast, or gamma, which accordingly makes them even less suitable than ordinary plates for astronomical use. On the other hand photographs taken on yellow-dyed film in a motion-picture camera having a good lens have shown greatly improved definition over those taken on undyed film. The principles enumerated above thus appear to be well founded and agree with practical findings. Since the relative amount of inherent and optical turbidity cannot be specified in any given class of photographic work, it will always be necessary to make an experimental determination of the conditions giving maximum sharpness or resolving power.

It has been the aim of the writer to show that resolving power can be put to a certain extent upon a mathematical basis, and that

secondary effects can be explained in terms of four major factors, turbidity, contrast, size of grain, and graininess. It is clear that far from being the simple phenomenon imagined by Wadsworth, photographic resolving power draws upon the whole field of photographic theory and can be but imperfectly described even from the broadest viewpoint.

Astronomers often ask for a high-speed plate of fine grain. Is not the advantage of small size of grain overestimated? For instance, the best linear resolution of double stars obtainable with large telescopes under the most favorable conditions is about 60 μ. The linear resolution obtained in the laboratory for emulsions of coarse grain and high speed is 20 μ. The true resolving power of the plate is thus far ahead of that which can be obtained in work with large telescopes. It is quite certain that an increase in the speed of plates will increase the resolution obtained in the large astronomical telescope. It is not so clear that an increase in resolving power will work in the same direction to any appreciable extent. But it is at least true that increased speed should be attained without any loss of the resolving power which the plates now have. In other words, it is more profitable to work in the direction of attaining higher speed than attaining better resolving power.

The subject of resolving power can be partly treated in a descriptive and more popular manner which is not without helpfulness. Consider a pair of close lines being photographed with continually increasing times of exposure, starting from threshold. Resolution is not attained until a certain density of the image has been built up. When the exposure is relatively light only a few scattering grains are affected, so that there is little or no contrast between the image and the surrounding field fog. Two close lines under these circumstances will not be distinguishable. Let the exposure be now increased. The vacant spaces between the scattering grains, which lie at random where the image should be, begin to fill in with grains, the lines begin to outline themselves and finally, when a certain point has been reached, are clearly resolved. This is a description of what happens in the ideal case where there is no spreading or turbidity. In an actual emulsion, in the case of a

line or star image, spreading begins before any relatively high density is reached, so that before a fine line can outline itself with sufficient distinctness, spreading has commenced, filling in the space between the two close-lying lines to a greater or less extent, thus limiting the resolving power which can be attained. This process is observable and has been studied on ordinary lantern plates and on yellow-dyed lantern plates. In these two plates, the size of grain is the same, and yet there is a very great difference in resolving power, as already explained. The test-object was photographed on them in turn with increasing exposure-times. With the ordinary plate, by the time the fine lines were filled in to a sufficient density to outline them well, the intervening spaces were also filled in with grains to such an extent as to destroy the separation. This filling in is due to turbidity and is seen not to be connected with the size of grain.

Compare this with what happens with the yellow-dyed plate. With light exposures only a few scattering grains are visible and there is no appearance of an image. But with longer exposure the lines begin to fill in until they are finally well outlined, and yet the spaces between remain free from grains; in short, the image builds itself up without spreading. Accordingly, neglecting the factor of the size of grain we can say that what happens is this: in an emulsion of poor resolution by the time the image is built up to a sufficient density to outline it, the spreading is so extended as to destroy resolution through the filling in of the spaces between the lines; in an emulsion of good resolution the image is built up to a sufficient density to outline it before marked infection of the grains in the intervening spaces has taken place. Accordingly turbidity and contrast of an emulsion are measures of resolving power, a conclusion already reached by more mathematical reasoning.

Research Laboratory
Eastman Kodak Company
Rochester, N.Y.
April 1920

STUDIES BASED ON THE COLORS AND MAGNITUDES IN STELLAR CLUSTERS

EIGHTEENTH PAPER: THE PERIODS AND LIGHT-CURVES OF 26 CEPHEID VARIABLES IN MESSIER 72[1]

By HARLOW SHAPLEY AND MARY RITCHIE

ABSTRACT

Periods and light-curves of 26 Cepheid variables in Messier 72 have been determined from eighteen plates taken with the 60-inch reflector. The variables are found to be much alike. The periods range from 0.33 to 0.66 day, and the median photographic magnitudes from 16.4 to 16.9, the mean value for all being 16.8. The difference of magnitude between these variables and the 25 brightest stars is unusually small, only 0.94.

Globular cluster Messier 72.—The *photographic magnitudes of twenty-nine comparison stars* were determined by the polar-comparison method. From the median magnitude of the variables the *parallax* of the cluster is computed to be 0."000039.

1. *Introduction.*—Exploration of the extent of the sidereal universe requires particular knowledge of the most distant objects on record—of the positions in space of faint globular clusters, galactic clouds, and spiral nebulae. For a remote cluster, the Cepheid variables appear to afford the most definite quantitative measure of the position in space, the distances of such stars being uniquely defined by the period-luminosity curve.[2]

The precise determination of the distance of a remote cluster aids also in the correlation between its distance and apparent diameter; and in any cluster the determination of the mean median magnitude of the cluster[3] variables serves in correlating the magnitudes of its brightest stars with the magnitudes of its variables. Thus in two ways the investigation of variables, whenever they can be found, bears on the problem of the distances of all clusters.

[1] *Contributions from the Mount Wilson Observatory*, No. 195.

[2] *Mt. Wilson Contr.*, No. 151; *Astrophysical Journal*, **48**, 89, 1918.

[3] The name "cluster variable" is used exclusively for those variables of the so-called Cepheid type whose periods are shorter than one day.

In view of the significance of these general problems, the discovery and study of Cepheid and cluster variables in the faint globular clusters has been undertaken at Mount Wilson as a feature of the work on the structure of sidereal systems.[1]

Systematic investigations of the periods and light-curves of the variable stars in four bright globular clusters have been published by Professor Bailey in the *Harvard Annals*,[2] affording results that have been of much importance in our work on Cepheid variables and stellar clusters. Bailey's work is based upon long series of plates, some made throughout an interval of more than twenty years—a circumstance that in many cases permits high precision in determining the period and range of light variation and also yields some information as to secular changes in these quantities.

Researches on variations in period and light-curve are necessarily based on extensive material and can be most advantageously made in the case of the brighter clusters, for which observations are easily obtained and analyzed; accordingly, the study at Mount Wilson has included neither this problem of secular variations nor the derivation of precise light-curves, but rather has been specifically concerned with the following points: (1) the average median magnitude of the cluster variables; (2) the bearing of the new material on the period-luminosity relation whenever both long and short period Cepheids are found; (3) the relation of the median magnitude to the brightest stars in the cluster; (4) the correlation of length of period with type of variation, magnitude, range, and distribution in the cluster.

None of these problems requires high accuracy in periods or light-curves if the variables are numerous, and therefore short series of plates are sufficient, provided the observations are properly arranged. In the following discussion of Messier 72 three of the points enumerated above are satisfactorily treated on the basis of twenty photographs with the 60-inch reflector. The variables studied are by far the faintest and most distant for which periods and light-curves are known.

[1] Compare section 1, *Mt. Wilson Contr.*, No. 190; *Astrophysical Journal*, **52**, 73, 1920.

[2] Vol. **38**, 1902; **78**, 1913–1919.

2. *Earlier work.*—Messier 72 (N.G.C. 6981 = B.D. $-13°5783$) is a faint globular cluster in Capricornus with the equatorial and galactic co-ordinates, for 1900:

Right ascension	$=20^h48^m0$	Declination	$=-12°55'$
Galactic longitude	$=3°$	Galactic latitude	$=-34°$

The photographic magnitudes of its brightest stars were determined three years ago (*Mount Wilson Contributions*, No. 152), and the mean of the twenty-five brightest was found to be 15.86.[1]

TABLE I

LIST OF PLATES

Plate Number	Date	J.D. and G.M.T (Heliocentric)	Length of Exposure	Quality	Average Deviation (Pg. mag.)
		2420000+			
3874*......	1917, Aug. 13	1454.825	2^m, 2^m, 2^m	Good	....
3949*......	Sept. 9	1481.819	3 , 3 , 3	Good	....
4063.......	Oct. 12	1514.693	10	Fair	±0.06
4163.......	Nov. 10	1543.662	10	Fair	0.06
4975.......	1919, July 21	2161.882	10	Good	0.05
4977.......	July 21	2161.907	10	Very good	0.02
4979.......	July 21	2161.968	10	Fair	0.05
4985.......	July 22	2162.746	15	Fair	0.05
4990†......	July 22	2162.830	10	Very good	0.05
4992.......	July 22	2162.892	12	Very good	0.03
5005.......	July 22	2162.975	15	Poor	0.07
5015.......	July 23	2163.738	12	Good	0.04
5022.......	July 23	2163.828	10	Good	0.05
5024.......	July 23	2163.903	10	Fair	0.07
5032.......	July 24	2164.766	15	Good	0.05
5034.......	July 24	2164.880	12	Very good	0.04
5039.......	Aug. 22	2193.788	10	Fair	0.05
5044.......	Aug. 23	2194.708	12	Good	0.06
5051.......	Sept. 21	2223.648	10 , 10, 2	Poor	0.05
5052.......	Sept. 21	2223.687	8, 8, 8	Good	±0.05
				Mean.....	±0.05

*Used only for bright comparison stars. †Used for positions and reproduction.

The diameter of the cluster was estimated as 2′, 2′.2, and 2′.4 by Melotte, Miss Davis, and Shapley, respectively. On the basis of these measures of diameter and magnitude the parallax was placed at 0″000034.

3. *Observations.*—The photographs listed in Table I were made at the primary focus of the 60-inch reflector, the first four on

[1] Owing to a typographical error the printed value is 15.94.

Seed 27, the remainder on Seed 30 plates. Multiple exposures, indicated in the fourth column, show which plates were compared directly with the polar standards. The estimated quality of the plate, as recorded at the time of observation, is given verbally in the fifth column; in the sixth column the quality is shown numerically for each plate in the form of average deviations from the adopted means for the magnitudes of comparison stars. This average deviation is the mean of the residuals given in Table II, where it is tacitly assumed that the light of the comparison stars is constant.

4. *Comparison stars.*—The photographic magnitudes of the comparison stars were determined as follows: The polar-comparison plates gave preliminary values; these magnitudes were plotted for all plates against the scale readings, after the usual correction had been applied to the latter for distance from the center and for scale irregularities. For each comparison star the deviation from a smooth curve was then tabulated and the systematic deviation for each star (mean of residuals with regard to sign) was applied as a correction to the preliminary magnitude. The new magnitudes of the comparison stars were again plotted against the scale readings. The residuals from the resulting new curves are those appearing in Table II, which serves to show both the degree of constancy of the light for all comparison stars and the numerical quality of each plate. The magnitudes of the variables have been read from this second set of curves.

The foregoing method of reduction is equivalent (1) to determining the zero point and scale of magnitudes by means of the four polar comparisons, (2) reducing the accidental error in the mean magnitude of a comparison star with data from twenty plates, and (3) deriving each magnitude of a variable star through indirect comparison with twenty-nine stars of sensibly constant light.

Table III gives differential positions and adopted magnitudes of the comparison stars, as well as the average deviation from the adopted means and the final systematic deviations. The last, which are in thousandths of a magnitude, have not been applied to the adopted magnitudes. It was noticed, after the magnitudes

TABLE II

Residuals for Comparison Stars

Comparison Stars

Plate Number	*a*	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*k*	*l*	*m*	*n*	*o*
4063	−10	+1	−2	0	−10	+10	0	+3	−2	+4	−11	−8	−5
4163	−3	0	−9	0	+6	−1	+5	−7	+4	+4	−6	+4	−4
4975	+4	−1	−1	+2	+7	−11	+6	0	+3	+5	+3	+1	−4
4977	0	−1	0	−3	−4	+3	−2	0	+1	−3	−2	−5	−10
4979	−10	−3	−4	+4	−2	+2	−3	+11	−1	+4	+10	+5	−17
4985	0	+2	+1	+1	+3	+5	0	+6	+8	−12	−6	+1	0
4990	−1	+4	+3	−3	−9	+9	+6	+2	+2	+1	−2	−8	+2
4992	+5	+1	+2	0	+1	−1	−1	−12	+1	−1	+5	0	+3
5005	−6	−4	+6	+9	+8	−2	+1	−2	−8	−2	−18	+5	+10
5015	+6	+3	−1	0	+2	+1	−12	+3	−6	....	+2	−3	+7
5022	+5	−1	0	−9	+10	+10	−10	+3	−1	+2	+3	−1	−3
5024	0	+2	+2	−5	−10	+5	−4	+4	+17	−13	−1	+1	+9
5032	+1	+3	0	+3	+8	−15	−2	+7	+5	−1	+12	+2	−10
5034	−2	0	−3	+3	−3	+8	−4	+4	0	−9	−2	0	−17
5039	+2	+1	−3	+6	−7	0	+1	+4	−1	+9	+3	+7	+5
5044	−7	−5	0	+2	+7	+5	+2	+4	−24	+8	−12	−6	+10
5051	+5	+4	+5	−7	0	−8	−3	−4	+3	−10	+7	0	−6
5052	+5	−2	+1	0	+3	−6	0	−3	+3	−6	+12	+8	−5

Plate Number	*p*	*q*	*r*	*s*	*t*	*u*	*v*	*w*	*x*	*y*	*z*	*α*	*β*	*γ*	*δ*	*ε*
4063	−9	+5	−4	−5	+9	−6	−8	−10	−9	0	+3	+13	+14	−3	+20	0
4163	−3	−5	0	+9	+7	−5	−14	−12	−8	−12	+6	−5	−2	−4	+17	0
4975	0	+11	−2	−2	−9	−2	−14	−7	+6	+13	−7	−7	+9	−10	+5	0
4977	0	−3	+6	+1	−5	−2	0	+1	+3	+3	+2	+3	0	0	−5	0
4979	−7	−10	−6	+5	0	−7	0	+5	+1	+9	+1	−7	−2	−3	+4	+7
4985	−13	+2	+2	−11	+5	−6	−6	−4	−5	−16	+7	+5	−1	+6	−8	0
4990	+1	−14	−9	−5	−3	−2	−15	+14	+2	+2	+7	+5	0	+3	−9	+10
4992	+2	+6	+1	−3	+3	−6	−2	+8	+5	0	−3	−1	+1	−4	0	−6
5005	+6	+10	+7	−10	−3	+7	−13	+5	−10	−8	+16	−11	+4	−1	+4	−6
5015	−5	−1	−6	+4	0	+13	0	0	+7	−4	−19	+1	−3	+6	−6	0
5022	−2	−5	−7	+7	+6	−15	0	+4	−3	−5	0	+7	−4	0	−6	+5
5024	−16	+1	0	+5	−23	−12	0	+12	+13	−3	+1	+8	−6	+12	−14	0
5032	−1	−5	+2	+8	−15	+4	0	−3	0	+4	−5	+5	−17	+6	−4	−6
5034	+1	+1	+4	+7	0	−5	+6	0	+6	−5	+1	+1	−6	+11	0	+11
5039	−1	+6	0	+5	+2	−7	−5	−7	−3	+13	−8	−18	+15	−2	−4	0
5044	+8	−10	−3	+5	0	+9	−8	0	+6	−6	−2	+2	+9	+10	+1	+7
5051	+10	+2	−3	−8	....	−1	....	+3	+4	+2	−8	+8	+9	−14	+2	−12
5052	+2	+2	+3	+8	−10	0	+4	−11	−10	−9	+4	−12	0	+8	−3	0

of the variables had been derived, that the final systematic deviations are negative for all comparison stars fainter than 17.2. This indicates a slight systematic error in drawing the curves; to correct for it, in making Table V we have applied +0.02 to the

TABLE III

COMPARISON STARS IN MESSIER 72

Star	Position		Photographic Magnitude	Average Deviation	Systematic Deviation
	x	y			
a	+0′40″.5	−1′11″.4	16.08	±0ᴹ.04	−0ᴹ.003
b	−1 7.5	+0 9.0	16.09	0.02	+0.002
c	+0 43.5	+0 14.4	16.16	0.02	−0.002
d	−0 28.5	+1 14.4	16.13	0.03	+0.002
e	−0 47.4	−0 51.0	16.67	0.06	+0.006
f	+2 7.5	+2 15.6	16.65	0.06	+0.008
g	+1 24.6	+0 43.5	17.31	0.03	−0.011
h	−0 35.4	+2 3.0	17.01	0.04	+0.013
k	−1 17.4	+1 12.6	17.18	0.05	+0.002
l	−2 18.6	+0 2.4	17.40	0.06	−0.012
m	−1 17.4	−1 14.4	17.14	0.06	−0.002
n	+1 27.6	−1 17.4	17.18	0.04	+0.002
o	+1 12.0	−0 19.5	17.51	0.07	−0.019
p	+0 58.5	−0 15.6	17.31	0.05	−0.015
q	+1 46.5	−0 3.6	17.27	0.06	−0.004
r	+0 31.5	+1 10.5	17.31	0.04	−0.008
s	+0 39.6	+1 20.4	17.00	0.06	+0.011
t	+0 14.4	+1 36.0	17.56	0.06	−0.021
u	+1 9.0	+0 8.4	17.38	0.06	−0.024
v	−1 48.0	−0 52.5	17.80	0.06	−0.044
w	−1 9.6	+0 47.4	17.20	0.06	−0.001
x	−1 21.0	+0 21.6	17.03	0.06	+0.003
y	−0 39.0	+0 51.0	17.09	0.06	−0.012
z	+0 33.6	+0 49.5	17.31	0.06	−0.002
α	−0 42.0	+1 42.6	17.27	0.07	−0.002
β	−0 46.5	+1 21.0	16.81	0.06	+0.011
γ	+2 4.5	−0 5.4	16.86	0.06	+0.012
δ	−0 50.4	+0 13.5	16.22	0.06	−0.003
ε	−1 41.4	+0 35.4	15.77	±0.04	+0.006
A*	+4 25.5	−1 54.0	....	....	....
				Mean ±0.05	

*Star A is B.D.−13°5785 (9ᴹ.1) = A.G. Camb. 7396.

magnitudes derived for the minima from analysis of the observations in Table IV.

5. *Magnitudes and light-elements of the variable stars.*—A photographic chart and the differential positions of the thirty-four stars in Messier 72, listed as probably variable by Miss Ritchie, are

TABLE IV

PHOTOGRAPHIC OBSERVATIONS OF VARIABLES IN MESSIER 72

J.D. AND G.H.M.T.	MAGNITUDE	PHASE	RESIDUAL	MAGNITUDE	PHASE	RESIDUAL	MAGNITUDE	PHASE	RESIDUAL
	No. 1. Period 0.61974			No. 2. Period 0.46561			No. 3. Period 0.48965		
2420000+									
1514.693	16.61	0.591	0	16.00	0.005	0	16.56	0.022	0
1543.662	17.25	.432	+3	16.87	.106	−1	16.99	.101	0
2161.882	16.90	.152	−2	16.04	.462	+2	17.33	.383	0
.907	16.94	.177	−2	16.16	.021	+7	17.33	.408	0
.968	17.07	.238	+2	16.77	.082	+4	16.77	.469	0
2162.746	17.26	.396	+5	17.29	.395	0	17.23	.268	0
.830	17.22	.480	−2	15.99	.013	−3	17.42	.352	+11
.892	17.26	.542	+1	16.42	.075	−22	17.35	.414	+2
.975	16.50	.005	+8	16.88	.158	−22	16.38	.007	+5
2163.738	16.93	.148	+2	16.09	.455	−4	17.16	.280	−8
.828	16.91	.238	−14	16.68	.080	0	17.34	.370	+1
.903	17.20	.313	+6	17.17	.155	+7	17.28	.445	−4
2164.766	16.60	.557	−50	16.86	.087	+9	17.25	.329	−4
.880	16.58	.051	−8	17.18	.201	−2	17.20	.443	−13
2193.788	17.23	.451	0	17.50	.241	+24	17.25	.462	+24
2194.708	16.87	.131	0	17.38	.230	+14	17.31	.402	−3
2223.648	17.04	.563	0	17.21	.302	−8	17.21	.453	−6
.687	16.78	.602	+29	17.24	.341	−6	16.19	.002	−7
	No. 4. Period 0.3619			No. 5. Period 0.4991			No. 7. Period 0.52463		
1514.693	17.22	.318	+11	17.32	.284	−8	17.19	.289	0
1543.662	16.71	.335	+4	17.40	.305	−1	17.14	.403	−11
2161.882	16.75	.068	−14	17.21	.140	0	16.79	.085	−7
.907	16.87	.093	−11	17.22	.165	−6	17.06	.110	+8
.968	17.11	.154	−8	17.43	.226	+5	17.11	.171	0
2162.746	17.32	.208	+4	16.54	.006	+4	17.34	.424	+8
.830	17.35	.292	+3	17.22	.090	+19	16.44	.508	−7
.892	16.26	.354	+7	17.35	.152	+11	16.60	.045	0
.975	16.68	.075	−16	17.43	.235	+4	17.02	.128	−2
2163.738	17.23	.114	+16	16.39	.000	0	17.26	.367	+2
.828	17.22	.204	−5	16.84	.090	−19	17.19	.457	−8
.903	17.28	.279	−2	17.28	.165	0	16.24	.007	0
2164.766	17.22	.056	+53	16.77	.030	+3	17.25	.345	+3
.880	17.18	.170	−8	17.14	.144	−8	17.36	.459	+9
2193.788	17.00	.126	−12	16.97	.104	−11	16.59	.513	+18
2194.708	17.36	.323	+42	16.71	.026	−1	17.20	.384	−4
2223.648	16.65	.311	−60	16.65	.018	0	17.27	.469	+2
.687	16.26	0.350	−4	16.99	0.057	+11	16.40	0.508	−11

	No. 8. Period 0.5743			No. 9. Period 0.5902			No. 10. Period 0.5483		
1514.693	16.76	0.391	−55	17.06	0.123	+17	16.99	0.250	−18
1543.662	16.77	.071	0	16.96	.172	− 6	16.87	.159	− 9
2161.882	17.35	.344	+ 9	17.33	.452	+ 3	17.28	.445	+10
.907	17.27	.369	0	17.22	.477	−10	16.80	.470	−14
.968	17.28	.430	− 1	16.85	.538	− 6	16.43	.531	+10
2162.746	16.69	.060	+ 2	17.03	.136	+10	17.00	.213	−10
.830	16.93	.144	−20	17.12	.220	+ 2	17.29	.297	+ 5
.892	17.19	.206	0	17.24	.282	+ 8	17.22	.359	− 6
.975	17.10	.289	−14	17.22	.365	0	17.33	.442	+14
2163.738	17.16	.477	− 8	17.00	.538	+ 7	16.70	.108	− 8
.828	16.39	.567	− 3	16.57	.038	+ 0	17.12	.198	+ 5
.903	16.66	.068	− 8	16.69	.113	−16	17.20	.273	− 1
2164.766	17.18	.357	− 8	17.18	.385	− 6	16.50	.039	+ 3
.880	17.32	.471	+ 5	17.34	.499	+ 2	16.98	.153	+ 4
2193.788	17.22	.089	+29	17.35	.488	+ 3	16.24	.001	0
2194.708	17.38	.435	+10	17.04	.227	− 7	17.22	.175	+21
2223.648	16.99	.086	+ 9	17.09	.237	− 3	17.14	.253	− 7
.687	17.24	.125	+15	17.11	.276	− 5	17.24	.292	+ 1
	No. 11. Period 0.3345			**No. 12. Period 0.4111**			**No. 13. Period 0.54182**		
1514.693	17.13	.218	−11	17.06	.331	− 4	16.82	.261	0
1543.662	16.89	.086	+ 3	16.93	.112	0	16.87	.513	+ 2
2161.882	17.24	.150	+11	16.75	.038	+22	16.75	.517	− 2
.907	17.33	.175	+14	16.59	.063	−12	16.00	.000	−10
.968	17.25	.236	− 1	17.11	.124	+14	16.41	.061	+ 3
2162.746	16.41	.010	0	16.72	.079	− 8	16.97	.297	+10
.830	17.09	.094	+19	16.98	.163	− 9	16.89	.381	−13
.892	17.19	.156	+ 5	17.26	.225	+12	17.10	.443	− 2
.975	17.41	.239	+14	17.29	.308	+15	16.56	.526	0
2163.738	16.37	.333	0	16.95	.249	−20	16.70	.206	− 1
.828	16.78	.087	− 8	16.96	.339	− 9	16.76	.296	−11
.903	17.00	.163	−15	16.32	.003	0	16.94	.371	− 5
2164.766	16.54	.023	+ 3	16.63	.044	+ 5	16.54	.150	− 5
.880	16.85	.137	−23	17.09	.158	+ 2	16.90	.264	+ 8
2193.788	17.32	.277	+ 2	16.97	.289	−19	17.22	.455	+ 8
2194.708	17.14	.194	− 7	16.66	.387	0	16.96	.292	+ 9
2223.648	16.51	.033	− 6	16.99	.138	− 2	16.78	.515	− 1
.687	16.82	0.072	+ 2	17.24	0.177	+14	16.19	0.013	+ 2

TABLE IV—*Continued*

J.D. and G.H.M.T.	Magnitude	Phase	Residual	Magnitude	Phase	Residual	Magnitude	Phase	Residual
	No. 14. Period 0.5904			No. 15. Period 0.5499			No. 16. Period 0.5641		
2420000+									
1514.693	17.10	0.233	+8	17.10	0.295	−7	17.13	0.143	+5
1543.662	16.96	.272	−6	16.87	.119	+17	17.35	.320	+1
2161.882	17.16	.342	+13	16.97	.252	−13	17.44	.308	+10
.907	17.03	.367	0	17.22	.277	+7	17.30	.333	−5
.968	17.00	.428	−3	17.36	.338	+13	17.32	.394	−1
2162.746	16.51	.026	0	16.25	.016	0	16.72	.044	+17
.830	16.68	.110	−18	16.48	.100	−16	16.98	.128	−5
.892	16.98	.172	0	16.77	.162	−8	17.26	.190	+6
.975	17.04	.255	+2	17.12	.245	+4	17.35	.273	+3
2163.738	16.93	.428	−10	17.13	.458	−15	17.26	.472	−2
.828	16.86	.518	0	16.16	.548	0	16.33	.562	0
.903	16.40	.003	0	16.32	.073	−15	16.84	.073	+6
2164.766	17.07	.276	+5	17.22	.386	−5	17.12	.372	−22
.880	17.06	.390	+3	17.04	.500	+5	17.28	.486	+3
2193.788	16.90	.368	−13	17.13	.263	+1	16.88	.061	+18
2194.708	17.04	.107	+18	16.76	.084	+18	17.17	.417	−20
2223.648	16.81	.118	−8	17.27	.429	0	16.37	.023	−2
.687	17.06	.157	+10	17.42	.468	+15	16.54	.062	−16
	No. 17. Period 0.56308			No. 18. Period 0.52016			No. 20. Period 0.59555		
1514.693	17.16	.516	−3	16.28	.453	+12	17.29	.327	+3
1543.662	17.19	.205	+5	16.40	.293	+12	16.93	.114	0
2161.882	17.04	.163	−2	16.08	.042	+17	17.12	.153	+7
.907	17.15	.188	+4	16.04	.067	+2	17.00	.178	−12
.968	17.25	.249	+4	16.12	.128	−5	17.25	.239	+5
2162.746	17.34	.464	+4	16.28	.386	0	17.29	.422	−4
.830	16.70	.548	+1	15.93	.470	−12	17.45	.506	+6
.892	16.68	.047	+8	15.77	.012	+2	17.17	.568	+17
.975	16.78	.130	−18	15.90	.095	−20	16.59	.055	−6
2163.738	17.20	.330	−7	16.32	.338	+4	17.13	.222	−5
.828	17.19	.420	−11	16.16	.428	−10	17.24	.312	−1
.903	17.32	.495	+2	15.79	.503	0	17.28	.387	−2
2164.766	17.15	.232	−4	16.21	.326	−7	16.79	.059	+12
.880	17.23	.346	−5	16.13	.440	−11	16.98	.173	−13
2193.788	17.02	.537	+5	16.18	.219	−7	17.35	.495	−3
2194.708	17.41	.331	+14	16.37	.098	+27	17.14	.224	−4
2223.648	16.39	.553	−3	16.15	.429	−13	16.61	.578	−5
.687	16.48	0.029	0	16.14	0.468	+7	16.48	0.021	+1

	No. 21. Period 0.5310			No. 23. Period 0.5969			No. 24. Period 0.4973		
1514.693	17.10	0.461	− 4	17.06	0.220	− 4	16.54	0.258	− 1
1543.662	17.25	.225	− 4	16.66	.538	−10	16.50	.383	+ 8
2161.882	17.30	.361	− 4	17.24	.370	+ 2	16.24	.459	− 4
.907	17.30	.386	− 4	17.16	.395	− 7	16.13	.484	−10
.968	17.36	.447	+ 4	17.22	.456	− 1	16.22	.048	− 7
2162.746	17.11	.163	− 5	16.30	.040	−12	16.54	.329	+ 5
.830	17.32	.247	+ 1	16.76	.124	−14	16.40	.413	+ 2
.892	17.35	.309	+ 2	17.05	.186	0	16.26	.475	+ 1
.975	17.33	.392	− 1	17.17	.269	0	16.26	.060	− 6
2163.738	16.85	.093	− 7	17.26	.435	+ 4	16.41	.326	− 9
.828	17.22	.183	0	17.19	.525	0	16.31	.416	− 6
.903	17.32	.258	0	16.18	.003	− 2	16.21	.491	0
2164.766	16.99	.059	+26	17.18	.269	+ 2	16.24	.360	−22
.880	17.18	.173	− 2	17.20	.383	− 2	16.28	.474	+ 3
2193.788	17.35	.407	+ 1	16.64	.043	+19	16.36	.041	+ 8
2194.708	17.36	.265	+ 4	17.29	.366	+ 7	16.41	.464	+14
2223.648	16.32	.000	0	16.58	.058	0	16.70	.063	+37
.687	16.52	.039	− 9	16.99	.097	+18	16.42	.102	+ 1

	No. 27. Period 0.65885			No. 28. Period 0.3638			No. 29. Period 0.36865		
1514.693	15.88	.020	+ 1	16.76	.045	+14	17.10	.212	−10
1543.662	15.72	.000	0	16.75	.273	−41	16.84	.058	+ 9
2161.882	16.83	.219	+ 7	16.63	.033	+ 8	16.83	.052	+11
.907	16.78	.244	− 4	16.54	.058	−13	16.87	.077	0
.968	16.75	.305	−17	16.82	.119	− 8	17.07	.138	+ 1
2162.746	17.08	.424	0	17.08	.170	+ 2	17.11	.179	− 3
.830	17.22	.508	+ 9	17.19	.254	0	17.28	.263	0
.892	17.05	.570	− 6	16.71	.316	− 1	17.35	.325	0
.975	15.84	.653	0	16.50	.035	− 6	16.50	.039	−11
2163.738	16.37	.098	+ 7	16.85	.070	+13	16.93	.065	+13
.828	16.68	.188	+ 1	17.12	.160	+ 9	17.14	.155	+ 4
.903	16.91	.263	+ 6	17.17	.235	0	17.32	.230	+ 9
2164.766	17.02	.467	−10	16.84	.007	+37	16.77	.355	0
.880	17.12	.581	+ 7	16.85	.121	− 5	16.79	.101	−17
2193.788	17.17	.500	+ 4	17.11	.289	+ 5	17.23	.254	− 4
2194.708	16.26	.102	− 6	16.69	.117	−19	16.79	.068	− 2
2223.648	16.05	.053	− 1	16.70	.317	0	17.29	.253	+ 2
.687	16.26	0.092	− 1	16.52	0.356	+ 3	17.30	0.292	− 2

TABLE IV—*Continued*

J.D. and G.H.M.T.	Magnitude	Phase	Residual	Magnitude	Phase	Residual	Magnitudes	
	No. 31. Period 0.55465			No. 32. Period 0.50544			No. 6	No. 19
2420000+								
1514.693	17.03	0.504	−2	16.94	0.447	0	16.99	17.26
1543.662	16.89	.077	−1	16.84	.107	−13	16.87	17.37
2161.882	17.14	.417	−3	17.18	.174	+6	16.97	17.25
.907	17.16	.442	+2	17.12	.199	−2	16.94	17.36
.968	16.98	.503	−7	17.16	.260	0	17.11	17.28
2162.746	17.13	.171	0	16.60	.027	−4	17.00	17.23
.830	17.12	.255	−8	16.93	.111	−5	16.93	17.35
.892	17.26	.317	+6	17.22	.173	+10	17.03	17.27
.975	17.15	.400	−3	17.04	.256	−12	16.81	17.07
2163.738	16.77	.054	0	16.53	.008	0	16.79	17.16
.828	16.96	.144	−12	16.91	.098	−3	16.96	17.28
.903	17.32	.219	+13	17.13	.173	+1	16.87	17.20
2164.766	16.82	.527	0	16.77	.025	+16	16.94	17.42
.880	16.96	.087	+1	16.96	.139	−15	16.98	17.26
2193.788	17.13	.153	+3	17.17	.237	+1	17.15	17.22
2194.708	16.98	.518	+5	17.25	.146	+17	16.98	17.22
2223.648	16.92	.062	+10	17.12	.276	−5	17.14	17.35
.687	17.06	0.101	+6	17.27	0.315	+10	16.80	17.36

J.D. and G.H.M.T.	Magnitudes							
	No. 22	No. 25	No. 26	No. 30	No. 33	No. 34		
1514.693	17.19	16.99	17.03	16.79	17.06	16.08	....	....
1543.662	17.04	16.96	16.93	16.81	17.19	16.43	....	....
2161.882	17.16	16.56	17.25	16.66	17.16	16.66	....	....
.907	17.19	16.45	17.10	16.59	17.14	16.37	....	....
.968	17.22	16.52	17.16	16.75	17.25	16.73	....	....
2162.746	17.13	16.91	17.08	16.49	17.13	16.37	....	....
.830	17.02	16.98	17.26	16.53	17.19	16.06	....	....
.892	17.26	16.60	17.17	16.38	17.08	16.38	....	....
.975	16.97	16.56	17.04	16.44	16.97	16.56	....	....
2163.738	17.20	16.63	17.16	16.91	17.23	16.41	....	....
.828	17.12	16.81	17.03	16.44	17.05	16.26	....	....
.903	17.23	17.04	17.04	16.53	17.17	16.63	....	....
2164.766	17.12	16.72	17.15	16.56	17.07	16.21	....	....
.880	17.18	17.01	17.14	16.72	17.14	16.13	....	....
2193.788	17.17	16.85	17.22	16.56	17.23	16.24	....	....
2194.708	17.04	16.69	17.17	16.87	17.06	16.19	....	....
2223.648	17.19	17.04	17.09	16.61	17.25	16.24	....	....
.687	17.33	16.78	17.08	16.73	16.84	16.40	....	....

given in the first section of ***Mount Wilson Contributions***, No. 190. The photographic magnitudes from measures by Miss Ritchie, the phases, and the residuals (in hundredths of a magnitude) from smooth mean curves, are given in Table IV. Table V contains the elements of light-variation and other data appropriate for statistical discussion, and is followed by notes on special cases. Except for a few instances cited in the notes, the periods are accurate to the fourth place of decimals. Four representative light-curves are shown in Figure 1. The determination of the periods and light-curves has been largely in the hands of Miss Mayberry.

6. *Discussion of results.*—Numbers 6, 19, 22, 26, and 33 do not appear to be conspicuously variable; Nos. 25, 30, and 34 undoubtedly vary, but it has not been possible to obtain uniform periods for them. For Nos. 18, 24, and 28 the periods are unsatisfactory to the extent stated in the notes to Table V. Twenty-two of the periods and light-curves, however, are sufficiently accurate for the summaries in Tables VI, VII, and VIII, which are arranged respectively with the arguments of period, median magnitude, and range.

Variable No. 27 was omitted from the correlation tables because of its deviation in period, distance from the center, range, and median magnitude from all of the twenty-two other variables with well-determined elements. Possibly it is an isolated variable, only by chance in the direction of Messier 72; but in that case it would be much the most distant isolated star on record, and some 15,000 parsecs from the galactic plane. Number 2, which has the second largest range of variation, is the second most distant from the center, a fact which suggests that No. 27 is probably a member of the cluster notwithstanding its somewhat anomalous character.

From Tables VI, VII, and VIII no definite correlations are evident, except that small range is associated with faint median magnitude—probably an indication merely that the maximum is frequently recorded too faint because of the insufficiency of observations near zero phase. The principal conclusion from these tables is that all the stars are much alike, particularly in median and minimum magnitudes.

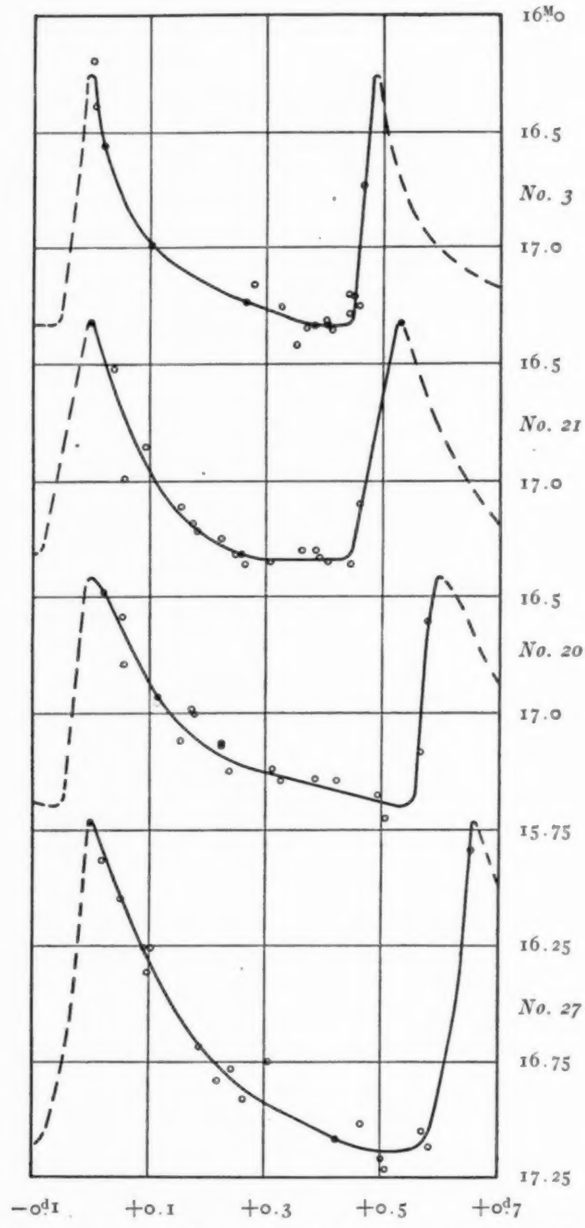


FIG. 1.—Light-curves of cluster variables in Messier 72. Ordinates are photographic magnitudes; abscissae phases from maximum in days.

TABLE V

ELEMENTS

Number of Variable	Distance from Center	Epoch of Heliocentric Maximum	Period	Photographic Magnitude			Range
				Maximum	Minimum	Median	
		2420000+					
1	1'.1	2162.97	0d.61974	16.40	17.27	16.84	0.87
2	3.6	2162.817	.46561	16.00	17.32	16.66	1.32
3	1.3	2162.968	.48965	16.25	17.35	16.80	1.10
4	1.8	2162.90	.3619	16.16	17.34	16.75	1.18
5	0.7	2163.738	.4991	16.40	17.43	16.91	1.03
7	0.9	2163.896	.52463	16.20	17.29	16.75	1.09
8	1.5	2163.835	.5743	16.40	17.32	16.86	0.92
9	0.9	2162.61	.5902	16.30	17.34	16.82	1.04
10	1.4	2163.63	.5483	16.23	17.32	16.77	1.09
11	1.1	2162.736	.3345	16.35	17.32	16.83	0.97
12	0.4	2163.90	.4111	16.31	17.17	16.74	0.86
13	0.4	2161.907	.54182	16.10	17.15	16.63	1.05
14	0.6	2163.90	.5904	16.40	17.06	16.73	0.66
15	1.1	2163.83	.5499	16.15	17.30	16.73	1.15
16	0.3	2163.83	.5641	16.30	17.37	16.83	1.07
17	0.7	2162.845	.56308	16.35	17.32	16.83	0.97
18	0.7	2162.88	.52016	15.70	16.28	15.99	0.58
20	0.9	2162.92	.59555	16.42	17.42	16.92	1.00
21	1.4	2162.583	.5310	16.32	17.37	16.85	1.05
23	2.5	2163.90	.5969	16.20	17.25	16.73	1.05
24	0.5	2161.92	.4973:	16.20	16.55	16.38	0.35
25	2.5	....	....	16.45	17.06	....	0.61
27	4.6	2162.981	.65885	15.72	17.15	16.43	1.43
28	1.7	2162.94	.36381	16.48	17.21	16.85	0.73
29	1.1	2161.83	.36865	16.40	17.37	16.87	0.97
30	2.0	....	....	16.38	16.91	....	0.53
31	0.6	2162.02	.55465	16.50	17.22	16.86	0.72
32	2.4	2163.73	0.50544	16.50	17.22	16.86	0.72
34	0.2	....	....	16.06	16.73	....	0.67

NOTES ON TABLE V

No.

3. The light-curve is shown in Figure 1.
5. The period 0d.5072 fits the observations nearly as well as the value in the table.
8. The adopted period is fairly satisfactory except for the unusually large residual for the first observation of the series.
9. The magnitude at maximum is uncertain.
18. This is the brightest variable in the cluster and has one of the smallest ranges of variation. The period is not quite satisfactory because of the large residuals and the gradual rise to maximum; it might be possible to represent the observations as well with a period slightly in excess of one day. If the star is double, as is suspected but not yet proved, the peculiarities of brightness and range are simply accounted for.
20. The light-curve is shown in Figure 1.
21. The light-curve is shown in Figure 1.

No.

24. Although this star is definitely variable, the range is so small that the adopted period cannot be accepted with certainty. The first observation of September 21, 1919, is very discordant.
25. The period appears to be some sub-multiple of one day.
27. The light-curve is shown in Figure 1; this variable has the longest period, the greatest distance, the greatest range, and, excepting No. 18, the brightest median magnitude.
28. The period is very uncertain.
30. The period may be something near 0^d46, but apparently the variation is irregular.
31. The magnitude at maximum is uncertain because of the lack of observations within forty minutes of zero phase.
34. The period is some sub-multiple of a day, but no satisfactory value has been found. This variable, which is the nearest of all to the center and was consequently measured with some difficulty, is brighter than the average and has a small range of variation. It may be a multiple star; cf. note on No. 18.

TABLE VI

Period	No.	Maximum	Minimum	Median	Range	Distance
$0^d.599$	5	$16^M.34$	$17^M.27$	$16^M.81$	$0^M.92$	$1'.2$
0.559	6	16.32	17.31	16.81	0.99	0.9
0.520	5	16.30	17.29	16.80	0.99	1.2
0.405	6	16.24	17.31	16.78	1.07	1.5

TABLE VII

Median Magnitude	No.	Period	Distance	Range
$16^M.69$	5	$0^d.549$	$1'.6$	$1^M.04$
16.77	6	0.488	1.1	1.06
16.83	5	0.522	0.9	0.99
16.88	6	0.516	1.2	0.89

TABLE VIII

Range	No.	Distance	Period	Median Magnitude
$1^M.17$	5	$1'.8$	$0^d.483$	$16^M.74$
1.06	6	1.1	0.558	16.77
0.99	5	0.9	0.472	16.87
0.79	6	1.1	0.543	16.81

All periods are shorter than a day. They show a considerable dispersion, but are not grouped into distinct classes as in clusters studied by Bailey. Thus we have:

Length of Period in Days	Number of Variables
<0.35	1
0.35–0.4	2
0.4 –0.45	1
0.45–0.5	3
0.5 –0.55	6
0.55–0.6	8
>0.6	1

The mean value of the period is $0^{d}517$.

The average value of the range in photographic magnitude is 0.99, with extremes, among the well-determined cases, of 1.43 and 0.66. The light-curves, so far as our observations are competent to say, all belong to the type usual in clusters, that is, to Bailey's subclass *a*, except that Nos. 12, 24, 25, and 34 might be assigned to subclass *c*, and 1, 20, and 27 to subclass *b*.

The median magnitude and its probable error is 16.80 ± 0.05, or, if No. 27 is included, 16.78 ± 0.09. The effect on the probable error justifies the exclusion of No. 27.

Using the methods outlined in *Mount Wilson Contributions*, No. 151, we derive for Messier 72 a parallax of $0''000039$, to which corresponds the space-position

$$R = 25.6$$
$$R \cos\beta = 21.2$$
$$R \sin\beta = -14.3,$$

where R is the distance in kiloparsecs and β is the galactic latitude. This value of the distance differs about 13 per cent from the value obtained in *Mount Wilson Contributions*, No. 151. The cause of the divergence can be traced to the unusually small difference of magnitude between the twenty-five brightest stars and the median ($m_{25} - med = 16.80 - 15.86 = 0.94$). For Messier 68, reported in *Mount Wilson Contributions*, No. 175, this same difference was also found to be smaller than the value adopted for earlier computations of the distances of clusters. A revision of the earlier work, based on a new value of $m_{25} - med$ may therefore be necessary, but apparently it would involve only a systematic decrease of the distances by 6 per cent or less.

Mount Wilson Observatory
August 1920

INFRA-RED SPECTRA OF ISOTOPES

By F. W. LOOMIS

ABSTRACT

Separation of the infra-red absorption doublets of HCl and HBr due to isotopes; theory.—If these bands are produced by the vibration of the nuclei along the line of centers of the molecules, the author shows that the frequencies should be approximately proportional to the square root of $(m_1+m_2)/m_1m_2$; hence the band lines due to HCl^{35} and HCl^{37} should differ by 1/1330, and those due to HBr^{79} and HBr^{81} should differ by 1/6478. In the case of the HCl band at 1.76μ, Imes found doublets whose components agree in separation and relative intensity with the theoretical results. In other cases the computed separations are less than the resolving power so far used.

Of the many attempts to find differences between the wave-lengths of spectrum lines of the isotopes of lead, two have been successful. Aronberg[1] found that the wave-length of the line at 4058 A was 0.0043 A greater for radioactive lead than for ordinary lead, though his probably too simple theory predicted a difference of only 0.00005 A or 1/80 as much as he observed. Merton, who had previously[2] failed to find any difference, has later[3] used a better method of comparison and has confirmed Aronberg's results. He also found a difference of 0.0022 A between the wave-length of this line for ordinary lead and for thorite lead, and a difference of 0.0055 A in the wave-lengths of the line λ 5350 for ordinary thallium and for thallium from pitchblende.

No theory has been published which accounts satisfactorily for even the order of magnitude of these differences, but it is probable that they are necessarily very small because the mass of the nucleus enters only through an expression such as

$$m=\frac{m_1m_2}{m_1+m_2},$$

[1] *Astrophysical Journal*, **47**, 96, 1918.

[2] *Royal Society Proceedings*, A, **91**, 198, 1915.

[3] *Ibid.*, A, **96**, 388, 1920.

where m_1 is the mass of the electron and m_2 that of the nucleus. Since $m_2/m_1 = 380{,}000$ for lead, the difference of one part in 207 in the atomic weights of the lead isotopes makes a very slight change in m and hence in the frequency. Much larger effects should be expected for any vibration in which two atoms or nuclei are the principal masses concerned, instead of one atom and an electron. Such a vibration would be in the infra-red, and probably, in the case of any compound of lead, so far in the infra-red as to be beyond the limits of known methods of measurement. But Aston has recently shown that chlorine[1] is a mixture of isotopes of atomic weights 35 and 37 with possibly a very little 39 and that bromine[2] is a mixture of isotopes of atomic weights 79 and 81. Both HCl and HBr show absorption bands in the infra-red, the structure of which has been approximately explained by Bjerrum[3] and later in such detail and so exactly by Kemble[4] as to leave little doubt that his picture of the mechanism is essentially correct and that the central frequency of the "fundamental" band (at $3.46\,\mu$ for HCl and $3.91\,\mu$ for HBr) is the frequency of vibration of the charged halogen atom and the hydrogen nucleus along their line of centers. These spectra therefore afford a promising opportunity to search for distinct lines due to different isotopes.

A sufficiently accurate value of the magnitude of the effect can be found by calculating the difference between the central frequencies corresponding to HCl^{35} and HCl^{37} or to HBr^{79} and HBr^{81} on the assumption that the amplitudes are infinitesimal.

Let m_1 be the mass of the hydrogen nucleus and m_2 that of the charged halogen atom. Then if x_1 and x_2 are the distances of their centers from some fixed point along the axis, the potential energy can be expressed as $\Phi(\zeta)$; where $\zeta = x_2 - x_1$. Let ζ_0 be the equilibrium position. Then $\Phi'(\zeta_0) = 0$. Define $\Phi(\zeta_0) = 0$. Then

$$m_1 \frac{d^2 x_1}{dt} = +\Phi'(\zeta)$$

[1] *Philosophical Magazine* (6), **39**, 611, 1920.

[2] *Nature*, **105**, 547, 1920.

[3] *Nernst Festschrift*, p. 90, 1912.

[4] *Physical Review* (2) **15**, 95, 1920.

and

$$m_2 \frac{d^2 x_2}{dt^2} = -\Phi'(\zeta) .$$

Or

$$m \frac{d^2 \zeta}{dt^2} = -\Phi'(\zeta)$$

where

$$m = \frac{m_1 m_2}{m_1 + m_2} .$$

Integrating:

$$\frac{d\zeta}{at} = \pm \sqrt{V_0^2 - \frac{2}{m}\Phi(\zeta)} .$$

Expanding $\Phi(\zeta)$ about ζ_0 we get, since

$$\Phi(\zeta_0) = \Phi'(\zeta_0) = 0$$

$$\Phi(\zeta) = \tfrac{1}{2}(\zeta - \zeta_0)^2 \Phi''(\zeta_0) + \tfrac{1}{6}(\zeta - \zeta_0)^3 \Phi'''(\zeta_0) + \ . \ . \ . \ .$$

and since, on the assumption that the amplitude is infinitesimal, we are justified in dropping all but the first term, we can integrate the expression for dt exactly and find for the period, T:

$$T = 4 \int_{\zeta_0}^{\zeta_1} \frac{d\zeta}{\sqrt{V_0^2 - (\zeta - \zeta_0)^2 \dfrac{\Phi''(\zeta_0)}{m}}} = 2\pi \sqrt{\frac{m}{\Phi''(\zeta_0)}}$$

or for the frequency,

$$\omega = \frac{1}{2\pi} \sqrt{\frac{\Phi''(\zeta_0)}{m}} .$$

Since $\Phi''(\zeta_0)$ depends only on the law of force between the particles which in turn presumably depends only on the charges and the arrangement of the electrons in the outer structure of the atoms and not in any way on the mass of the nuclei, it should be the same for the two isotopes and the difference between them should show itself only through the quantity

$$m = \frac{m_1 m_2}{m_1 + m_2} ,$$

the square root of which occurs in the denominator of the expression for the frequency. Denoting quantities relating to the lighter isotopes as m', m_2', etc., and to the heavier isotopes as m'', m_2'', etc., we have for chlorine:

$$m'=\tfrac{35}{36},\ m''=\tfrac{37}{38},\ \frac{m''}{m'}=1+\tfrac{2}{1330},$$

hence

$$\omega'-\omega''=\frac{\omega'}{1330}$$

and for bromine:

$$m'=\tfrac{79}{80},\ m''=\tfrac{81}{82},\ \frac{m''}{m'}=1+\tfrac{2}{6478},\ \omega'-\omega''=\frac{\omega'}{6478}$$

where m_1 has, for simplicity, been taken as 1. This does not affect the result.

The non-radiating states are supposed to have characteristic energies which are integral multiples of $h\omega$ and the frequency emitted or absorbed when the system changes from one of these non-radiating states to another is the difference in energy divided by h. Hence, on the simple assumption of infinitesimal amplitudes, or linear law of force, which we have made, the frequencies of the centers of the "fundamental" and "harmonic" bands will be ω', $2\omega'$, $3\omega'$, etc., for the lighter isotopes and ω'', $2\omega''$, $3\omega''$, etc., for the heavier. In each case the corresponding frequencies for the two isotopes will bear the same ratio: $1+1/1330$ for chlorine and $1+1/6478$ for bromine.

The structure of the bands has been most accurately observed by Imes[1] and is shown in Figures 1 to 4 taken from his paper. It has been accurately accounted for by Kemble[2] using the theorem, enunciated by Lord Rayleigh, that an oscillator vibrating with a frequency ν and rotating with a frequency ν_r absorbs or emits light of frequency $\nu \pm \nu_r$, and modifying the Bjerrum hypothesis in the manner required by the new Sommerfeld[3] formulation of the quantum theory for systems with more than one degree of freedom.

[1] *Astrophysical Journal*, **50**, 251, 1919.

[2] *Loc. cit.*

[3] *Annalen der Physik* (4) **51**, 1, 1916.

He shows that if the law of force were strictly linear the infra-red absorption spectrum lines would be given by

$$\nu = n\omega \pm p\nu_1$$

where

$$n = 0, 1, 2, 3, \ldots .$$
$$p = 1, 2, 3, 4, \ldots . .$$

Hence they have the general structure shown in the figures, pairs of lines grouped about a central frequency which is not represented by a line. (The asymmetry is quantitatively accounted for by non-linearity in the law of force.)

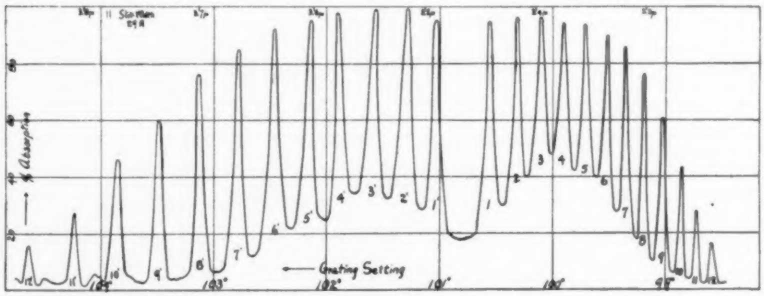


FIG. 1.—The HCl band at 3.46μ, mapped with 7500-line grating. HCl at atmospheric pressure (Imes, *Astrophysical Journal*, **50**, 260, 1919).

An examination[1] of the way the mass enters in the expression for $\nu_1 = \dfrac{h}{4\pi^2 m\zeta^2}$ and in Kemble's formula for the asymmetry constant shows that although the complete expression for the effect

[1] ν_1 varies as $1/m$, while A and ω_0' vary as $1/\sqrt{m}$. Hence, for chlorine,

$$\nu_1'^2 a'(\tau) - \nu_1''^2 a''(\tau) = \frac{m'' - m'}{m}[3.125 \times \tfrac{3}{2} - 0.638 \times \tfrac{5}{3}\tau] \times 6.255^2 \times 10^8$$

and

$$\nu_1' - \nu_1'' = \frac{m'' - m'}{m} \times 6.255 \times 10^{11}$$

where

$$\frac{m'' - m'}{m} = \tfrac{2}{1880} .$$

The equations also predict a change in the difference between the frequency of the "harmonic" and twice the frequency of the "fundamental" but as Kemble's equation is admittedly (*op. cit.*, p. 106) at fault here, predicting a much larger effect than

of difference of mass is more complex than the one used, the additional terms are quite negligible. This was to be expected, since m_2 enters nowhere except through m, and since all other frequencies and frequency differences are small compared with ν.

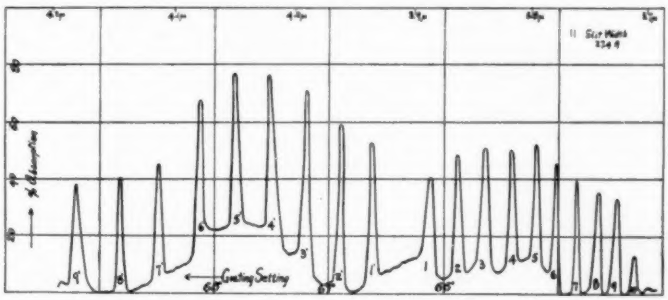


FIG. 2.—The HBr band at 3.91μ, mapped with 7500-line grating

Consequently the net difference between the spectra of isotopes will be that the wave-lengths of lines in the spectrum of the heavier isotope will be longer than the corresponding lines for the lighter isotope in the ratio $1+1/1330:1$ for chlorine and $1+1/6478:1$ for bromine. Since the average atomic weight of chlorine is 35.46 the amounts of Cl^{35} and Cl^{37} present in ordinary chlorine must be as $1.54:0.46$ or as $3.35:1$ and, if the lines were absolutely sharp and perfectly resolved, the absorption spectrum of ordinary HCl should consist of pairs of lines separated by $1/1330$ of their frequency and the one of shorter wave-length should have about 3.35 the intensity of the other. The average atomic weight of bromine is 79.92, hence the two isotopes are present in nearly equal proportions and the absorption spectrum of HBr should consist of pairs of lines of nearly equal intensity separated by

---

observed, it is best to ignore this point and consider $\omega(r, o)$ to vary as ω_0', hence as $1\sqrt{m}$. Then, for the fundamental,

$$\frac{\nu'-\nu'}{3\times 10^{10}}=2.17-0.0006\,p^2\pm 0.0314\,p$$

and for the harmonic

$$\frac{\nu'-\nu''}{3\times 10^{10}}=4.26-0.0003\,p^2\pm 0.0314\,p.$$

Since no lines have been observed for which $p>12$ we can neglect the terms in p and p^2, and thus find the results in Table I.

1/6478 of their frequency. Fluorine, atomic weight 19.00, has been shown by Aston[1] to be pure and the absorption spectrum of HF should therefore consist of single lines.

TABLE I

	ν	$\nu'-\nu''$	ν_1	$\nu'-\nu''/\nu_1$
HCl fundamental.......	2887	2.2	21	0.1
HCl harmonic..........	5667	4.3	21	0.2
HBr fundamental.......	2559	0.4	17	0.02
HBr harmonic..........	5030*	0.8	17	0.05

* Brinsmade and Kemble, *Proceedings National Academy of Sciences*, **3**, 422, 1917.

Table I shows in column 3 the differences of frequency $\nu'-\nu''$, calculated for the fundamentals and harmonics of HCl and HBr, in column 4 the average difference of frequency, ν_1, between the lines corresponding to successive rotational quanta and in column 5 the ratio $\nu'-\nu''/\nu_1$. All frequencies are in wave-numbers per cm.

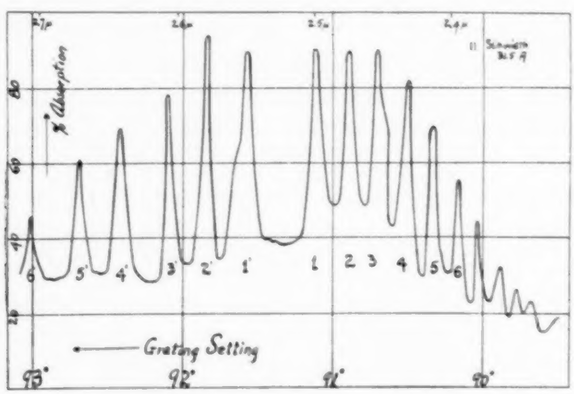


FIG. 3.—The HF band at 2.52μ, mapped with 7500-line grating

It is not surprising that no separate lines can be distinguished in the fundamentals of HCl (Fig. 1) and HBr (Fig. 2), as the expected separations are only $\frac{1}{10}$ and $\frac{1}{50}$ of the respective distances between rotational quantum lines. But each rotational quantum line in the first harmonic of the HCl spectrum (Fig. 4) shows on the long-wave side a distinct satellite of less intensity and

[1] *Nature*, **105**, 547, 1920.

separated from it by an average measured interval (see Table II), of 14A or $\nu'-\nu''=4.5$ wave-numbers, which agrees within the probable error of measurement with the calculated value in Table I. Clearly these are the predicted lines due to the heavier isotope.

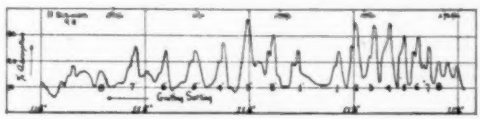


FIG. 4.—The HCl band at 1.76μ, mapped with 20,000-line grating. HCl at atmospheric pressure.

Table II shows the measured[1] values of the interval $\nu'-\nu''$ for each of the rotational quantum lines in the "harmonic" of HCl; the numbers of the lines, as in Figure 4, are counted from the center, the primed numbers being on the side of the long wave-length. The positions of the "secondary maxima" are taken as the points where the curve is estimated to lie farthest from its "undisturbed" position.

TABLE II*

Line	$\lambda''-\lambda'$	$\nu'-\nu''$	Line	$\lambda''-\lambda'$	$\nu'-\nu''$
1	12	3.9	1′	14	4.5
2	11	3.6	2′	16	5.1
3	13	4.3	3′	....	....
4	....	....	4′	10	3.1
5	14	4.6	5′	9	2.8
6	16	5.3	6′	22	6.7
7	16	5.3	7′	....	....
8	....	....	8′	....	....
			Average	14	4.5 ± 0.4

* λ in A, ν in wave-numbers per cm.

Imes noticed these satellites and before Aston had published his results, remarked:

> The apparent tendency of some of the maxima to resolve into doublets in the case of the HCl harmonic may be due to errors of observation, but it seems significant that the small secondary maxima are all on the long-wave side of the principal maxima which they accompany. It is, of course, possible that still higher dispersion applied to the problem may show even the present curves to be composite.[2]

[1] Measurements taken from the original curves, through the courtesy of Dr. Imes.

[2] *Op. cit.*, p. 275.

The harmonic of HBr was found by Brinsmade and Kemble,[1] but not even resolved into rotational quantum lines. It was not found by Imes and probably will not easily be resolved into lines of separate isotopes when it is found. The fundamental of HF (Fig. 3) shows, as expected, no satellites.

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September 1920

[1] *Op. cit.*

ELEMENTS OF THE ECLIPSING SYSTEM RT LACERTAE

By MARY FOWLER[1]

ABSTRACT

Eclipsing system RT Lacertae.—Luizet's observations, both 1910 and 1915, were reduced, a table of normals was prepared, the phases were slightly adjusted and two sets of *elements* were computed, (1) assuming central total eclipses of two equal elliptical stars and (2) assuming an obscuration of 0.990 (Table II). The *light-curve* at the minima is shown and a *diagram of the system*. In spite of the fact that the giant stars are less than a diameter apart, there is no evidence of an effect of radiation between them; on the contrary they seem actually less luminous at the limb than at the center. The system would well repay spectroscopic study.

In 1915 Shapley[2] published elements for this star ($\alpha = 21^h\ 57^m.4$, $\delta = +43°24'$ for 1900) which he derived from the combined observations of Luizet[3] and Enebo,[4] but inasmuch as Luizet published in the same year[5] additional data derived by adding new material to his observations, giving an excellent light-curve, it appeared worth while to make a new solution.

Luizet's observations are here reduced, using the period $5^d.073896$ derived by him, and the magnitudes obtained by changing Luizet's "degrees" to magnitudes, setting 15.48 degrees equal to magnitude 8.80 and allowing a difference of one magnitude to twelve degrees, as worked out by Shapley.

Normals were formed by combining those of Luizet in pairs, and an adjustment of the phases of minima by least squares indicated that a correction of $+0^d.013$ is to be applied to all the phases, and also that there is in addition a real displacement of the secondary minimum by $+0^d.022$. This shows that the orbit is

[1] The solution for this star was made by Miss Fowler while a member of the Princeton Observatory staff, and left by her at the beginning of the war almost ready for publication. At the request of Professor Russell, I have completed a few unfinished bits of computation and arranged the paper for publication.—Bancroft Walker Sitterly.

[2] *Contributions from the Princeton University Observatory*, No. 3.

[3] *Bulletin Astronomique*, **27**, 306, 1910.

[4] *Astronomische Nachrichten*, **184**, 396, 1910.

[5] *Bulletin Astronomique*, **32**, 68, 1915.

slightly eccentric, giving $e \cos \omega = +0.00128$. As far as can be determined from the observations, the durations of primary and secondary are equal; hence $e \sin \omega$ is sensibly zero (though far less accurately determined than $e \cos \omega$) and for the purpose of finding the elements of the system the orbit may be treated as circular.

The normals give an excellent light-curve, showing two well-defined minima with a combined loss of light exceeding unity, a

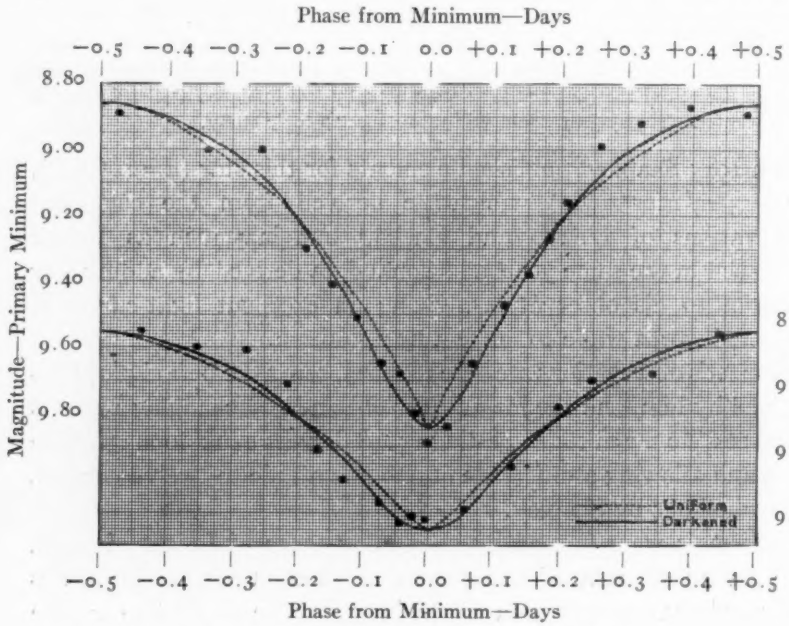

FIG. 1.—Light-curve of RT Lacertae at eclipses

condition suggesting total eclipses of two equal elliptical stars. Ellipticity of the stars is also indicated by the bowed form of the curve outside eclipse, without constant maximum; and the long duration of the eclipses compared with the total period, showing that the stars are close together, makes this condition very probable. Accordingly the ellipticity constant z was determined in the usual way from the light-curve outside minimum, and the light-curve "rectified," making the combined loss of light 0.990. Then two solutions were made, assuming stars of equal size, the first solution

TABLE I

Table of Observations, RT Lacertae

No.	Luizet's Phase	Adjusted Phase	Observed Magnitude	O−C Uniform	O−C Darkened (Adopted Curve)	O−C Uniform with 5 Per Cent Reflection
	Days	Days				
1	0.014	0.027	9.84	$+0^{M}.11$	$+0^{M}.03$	$+0^{M}.06$
2	0.055	0.068	9.65	+0.07	−0.02	+0.02
3	0.102	0.115	9.47	+0.03	−0.02	+0.02
4	0.138	0.151	9.38	+0.04	+0.01	+0.03
5	0.168	0.181	9.27	+0.02	0.00	+0.01
6	0.197	0.210	9.16	−0.04	−0.03	−0.03
7	0.247	0.260	8.99	−0.11	−0.08	−0.10
8	0.307	0.320	8.92	−0.08	−0.06	−0.08
9	0.382	0.395	8.87	−0.04	−0.03	−0.03
10	0.469	0.482	8.89	+0.03	+0.03	+0.03
11	0.599	0.612	8.83	....	−0.02	....
12	0.734	0.747	8.86	....	+0.02	....
13	0.935	0.948	8.82	....	−0.01	....
14	1.170	1.183	8.80	....	−0.02	....
15	1.291	1.304	8.81	....	−0.01	....
16	1.433	1.446	8.85	....	+0.03	....
17	1.552	1.565	8.84	....	+0.02	....
18	1.700	1.713	8.82	....	−0.02	....
19	1.853	1.866	8.87	....	+0.03	....
20	2.108	2.121	8.85	....	−0.02	....
21	2.194	2.207	8.90	−0.04	−0.02	−0.02
22	2.269	2.282	8.91	−0.11	−0.08	−0.10
23	2.331	2.344	9.01	−0.08	−0.07	−0.07
24	2.377	2.390	9.21	+0.06	+0.05	+0.06
25	2.417	2.430	9.30	+0.08	+0.06	+0.08
26	2.470	2.483	9.36	+0.04	0.00	+0.05
27	2.502	2.515	9.43	+0.05	+0.01	+0.06
28	2.521	2.534	9.41	−0.01	−0.03	0.00
29	2.545	2.558	9.42	−0.03	−0.03	−0.02
30	2.600	2.613	9.39	+0.03	−0.01	+0.05
31	2.672	2.685	9.26	+0.04	+0.02	+0.05
32	2.744	2.757	9.08	−0.03	−0.03	−0.01
33	2.793	2.806	9.00	−0.04	−0.03	−0.03
34	2.886	2.899	8.98	+0.03	+0.05	+0.05
35	2.987	3.000	8.86	−0.02	−0.01	−0.01
36	3.102	3.115	8.90	....	+0.05	....
37	3.179	3.192	8.87	....	+0.02	....
38	3.239	3.252	8.84	....	−0.01	....
39	3.375	3.388	8.85	....	+0.01	....
40	3.534	3.[illegible]47	8.84	....	+0.02	....

TABLE I—*Continued*

No.	Luizet's Phase	Adjusted Phase	Observed Magnitude	O−C Uniform	O−C Darkened (Adopted Curve)	O−C Uniform with 5 Per Cent Reflection
	Days	Days				
41	3.678	3.691	8.85	....	$+0^{M}.03$	....
42	3.823	3.836	8.82	....	0.00	....
43	3.951	3.964	8.82	....	0.00	....
44	4.052	4.065	8.81	....	−0.01	....
45	4.170	4.183	8.84	....	0.00	....
46	4.335	4.348	8.84	....	0.00	....
47	4.485	4.498	8.87	....	+0.02	....
48	4.589	4.602	8.89	+0.03	+0.03	+0.03
49	4.724	4.737	9.00	+0.01	+0.04	+0.03
50	4.805	4.818	9.00	−0.11	−0.08	−0.10
51	4.874	4.887	9.30	+0.04	+0.04	+0.04
52	4.914	4.927	9.41	+0.06	+0.03	+0.04
53	4.952	4.965	9.51	+0.05	0.00	+0.03
54	4.989	5.002	9.65	+0.09	−0.02	+0.04
55	5.017	5.030	9.68	+0.02	−0.08	−0.02
56	5.041	5.054	9.80	+0.04	−0.02	0.00
57	5.062	5.075	9.89	+0.05	+0.05	+0.04

Probable Error of one normal, on "darkened" hypothesis = $\pm 0^{M}.0236$.

on the "uniform" hypothesis, with central total eclipses, the second on the "darkened" hypothesis, with eclipses just failing to be total, the percentage of obscuration being 0.990. Table I gives the normals formed by pairing Luizet's observations, and the residuals for the two solutions, and Figure 1 gives the curves at the two minima. It is quite evident that the "darkened" curve gives much the better fit, for the other is obviously not steep enough during eclipse, points corresponding to small percentages of obscuration lying generally above the computed curve, while those near totality fall below the curve.

The curve between eclipses shows no evidence of an effect of radiation between the stars. But since by either of the solutions they are separated from one another by less than the diameter of one of them, some such effect might be expected, and Professor Russell suggested that an assumption of this nature might improve the "uniform" solution. Such a solution was therefore made, assuming that the side of each star toward the other was brightened by approximately 5 per cent of the system's total light—a value

much exceeding that found for any eclipsing system except TV Cassiopeiae, and hence a reasonable outside limit—but the curve so computed could not be made to fit the observations, however the constants were varied, as well as the "darkened" curve, though it was some improvement on the "uniform" curve, as shown by the residuals given in Table I with the others.

It appears then that the stars are actually less luminous at the limb than at the center. It is usually possible, even in such a case,

TABLE II

TABLE OF ELEMENTS, RT LACERTAE

EPOCH AND PERIOD = J.D. 2418024.444 G.M.T. $\pm 0^d0009 + (5^d073896 \pm 0^d00029)$ E

Element	Symbol	Value: Uniform	Value: Darkened
Ratio of radii of stars	k	1.000	1.000
Maximum percentage of eclipse	a_0	1.000	0.990
Semi-major axis	$a_b = a_f$	0.294	0.277
Semi-minor axis	$b_b = b_f$	0.262	0.267
Eccentricity of meridian section of stars	$\epsilon = \sqrt{z \operatorname{cosec}^2 i}$	0.334	0.264
Normal component of orbital eccentricity	$e \cos \omega$	+0.00128	
Inclination of orbit	i	90°0′	89°10′
Least apparent distance of centers	$\cos i$	0.0000	0.0146
Light of brighter star	L_b	0.585	
Light of fainter star	L_f	0.415	
Ratio of surface brightness of stars	J_b/J_f	1.41	
Density of stars in terms of sun	$\bar{\rho}_b = \rho_f$	0.013	
Light of brighter star in terms of sun	$\bar{L}_b$	15	
Greatest radius of stars in terms of sun	$\bar{a}_b = \bar{a}_f$	4.3	
Absolute magnitude of brighter star	M_b	+1.8	
Parallax	π	0″0029	
Distance in light-years		1100	
Spectral type		G_5	

to obtain a fairly close representation of the observations on the assumption of uniformly illuminated disks, by varying the elements, but in the present instance, where the observations clearly demand nearly total eclipses of nearly equal stars, this cannot be done, and distinction between the two hypotheses is definitely possible.

The adopted "darkened" curve for primary minimum is fitted to the secondary minimum by simply changing the light-intensity in the ratio of the depth of the secondary to the primary, and

displacing the curve in time by the amount indicated in the least-squares solution for $e \cos \omega$.

Table II gives elements for the system, computed on both "uniform" and "darkened" hypotheses, without including effect of radiation, which has no warrant from the actual observations. It is evident that for both hypotheses these elements are almost identical; the "darkened" stars are not so elliptical in figure as the others, and the "uniform" solution makes the orbit exactly edge-on to us, while the "darkened" orbit-plane makes a small angle, less than one degree, with the line of sight. The hypothetical

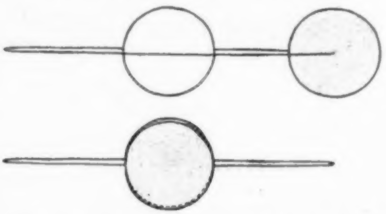

FIG. 2.—RT Lacertae at elongation and at primary minimum according to the "darkened" solution.

densities, dimensions, absolute magnitude, and parallax were computed by the method Shapley gives.[1] The surprisingly low density, taken with the spectral type, G5, shows this pair to be typical "giants" and a system which would well repay careful spectroscopic study with a view to determining absolute physical elements. The magnitude of the fainter star is about 9.9, which should put its spectrum within the reach of our more powerful spectrographs, and since the hypothetical dimensions of the system give at elongation a relative velocity in the line of sight of 150 km/sec., the spectra of the components should be separable even though the lines be broad and ill defined.

Figure 2 illustrates the system as seen from the earth.

PRINCETON UNIVERSITY OBSERVATORY
June 1920

[1] *Contributions from the Princeton University Observatory*, No. 3, pp. 9, 117.

PREPARATION OF ABSTRACTS

Every article in the *Astrophysical Journal*, however short, is to be preceded by an abstract prepared by the author and submitted by him with the manuscript. The abstract is intended to serve as an aid to the reader by furnishing an index and brief summary or preliminary survey of the contents of the article; it should also be suitable for reprinting in an abstract journal so as to make a reabstracting of the article unnecessary. Therefore, *the abstract should summarize the information completely and precisely*, and also, in order to enable a reader to tell at a glance what the article is about and to enable an efficient index of the subject-matter of the abstract to be readily prepared, *the abstract should contain a set of subtitles which together form a complete and precise index of the information contained in the article.*

In the preparation of abstracts, authors should be guided by the following rules, which are illustrated by the abstracts appearing in the *Astrophysical Journal* since January 1920.[1] First the new information contained in the article should be determined by a careful analysis; then the subtitles should be formulated; and finally the text should be written and checked.

RULES

1. *Material not new* need not be analyzed or described; a valuable summary of previous work, however, should be noted.

2. *The subtitles should together include all the new information;* that is, every measurement, observation, method, improvement of apparatus, suggestion, and theory which is presented by the author as new and of value in itself.

3. *Each subtitle should describe the corresponding information so precisely* that the chance of any investigator's being misled into thinking the article contains the particular information he desires when it does not, or vice versa, may be small. "Zeeman effect for metallic furnace spectra" is too broad unless all metals have been studied, for an investigator may be interested, at the time, in only one metal; but "Infra-red arc spectrum of iron to 3 μ" evidently satisfies this rule. It is particularly desirable that ranges of variation of temperature, wave-length, pressure, etc., be given.

4. *The text should summarize the author's conclusions and should transcribe all numerical results of general interest*, including all that might be looked for in a table of astronomical and physical constants, with an indication of the

[1] The rules and illustrative abstracts were prepared by G. S. Fulcher, while Scientific Associate with the National Research Council.

accuracy of each. It should give all the information that anyone, not a specialist in the particular field involved, might care to have in his notebook.

5. *The text should be divided into as many paragraphs* as there are distinct subjects concerning which information is given. Parts of subtitles may be scattered through the text but the subject of each paragraph must be indicated at the beginning.

6. *Complete sentences* should be used except in the case of subtitles. The abstract should be made as readable as the necessary brevity will permit.